

# An investigation of the tangent splash of a subplane of $\text{PG}(2, q^3)$

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## Abstract

In  $\text{PG}(2, q^3)$ , let  $\pi$  be a subplane of order  $q$  that is tangent to  $\ell_\infty$ . The tangent splash of  $\pi$  is defined to be the set of  $q^2 + 1$  points on  $\ell_\infty$  that lie on a line of  $\pi$ . This article investigates properties of the tangent splash. We prove results about sublines contained in a tangent splash, transitivity results, and counting results. We also show that a tangent splash is a Sherk surface. Further, in the Bruck-Bose representation of  $\text{PG}(2, q^3)$  in  $\text{PG}(6, q)$ , we prove the existence of a set of cover planes of a tangent splash, and investigate the tangent subspace of a point of  $\pi$ .

## 1 Introduction

In [2], the authors study the Bruck-Bose representation of  $\text{PG}(2, q^3)$  in  $\text{PG}(6, q)$  and determine the representation of order- $q$ -subplanes and order- $q$ -sublines of  $\text{PG}(2, q^3)$  in  $\text{PG}(6, q)$ . In this article we investigate in more detail the *tangent order- $q$ -subplanes* of  $\text{PG}(2, q^3)$ , that is, the subplanes of  $\text{PG}(2, q^3)$  that have order  $q$  and meet  $\ell_\infty$  in exactly one point.

Let  $\pi$  be a subplane of  $\text{PG}(2, q^3)$  of order  $q$  that meets  $\ell_\infty$  in 0 or 1 point. Each line of  $\pi$ , when extended to  $\text{PG}(2, q^3)$ , meets  $\ell_\infty$  in a point. The set of points of  $\ell_\infty$  that lie on a line of  $\pi$  is called the *splash* of  $\pi$ . If  $\pi$  is tangent to  $\ell_\infty$  at a point  $T$ , then the splash of

$\pi$  consists of  $T = \pi \cap \ell_\infty$ , and  $q^2$  further points. We call this splash a *tangent splash*  $\mathcal{S}_T$  with *centre*  $T$ . If  $\pi$  is disjoint from  $\ell_\infty$ , then the splash contains  $q^2 + q + 1$  points, and we call it an *exterior splash*. In this article we investigate the tangent splash of a tangent order- $q$ -subplane. In future work the authors investigate the exterior splash.

Section 2 contains the necessary background information. Section 3 investigates order- $q$ -sublines contained in a tangent splash, and this investigation is continued in Section 7. We show that the unique order- $q$ -subline of  $\ell_\infty$  containing the centre  $T$  and two further points of a tangent splash  $\mathcal{S}_T$  is contained in  $\mathcal{S}_T$ , and further, these are the only order- $q$ -sublines contained in  $\mathcal{S}_T$ .

In Section 4 we prove that the collineation group  $\text{PGL}(3, q^3)$  is transitive on the tangent splashes of  $\ell_\infty$ , and prove further useful group theoretic results about tangent splashes. Section 5 looks at counting the number of tangent splashes on  $\ell_\infty$ . Further, we look at how many points are needed to define a unique order- $q$ -subplane with a given tangent splash.

To continue with our investigation, we need to introduce coordinates, and in Section 6 we coordinatise an order- $q$ -subplane of  $\text{PG}(2, q^3)$ . In Section 8 we prove that a tangent splash is a Sherk surface as defined in [13].

Section 9 looks at a tangent splash in the Bruck-Bose representation in  $\text{PG}(6, q)$ . A tangent splash corresponds to a set of  $q^2 + 1$  planes contained in a regular spread. We show that there is a set of  $q^2 + q + 1$  *cover planes* that meet every element of a tangent splash, and are completely contained in the tangent splash.

In Section 10 we begin with a tangent order- $q$ -subplane  $\pi$ , and for each affine point  $P \in \pi$  we construct another order- $q$ -subplane  $P^\perp$ , called the tangent subspace, and investigate the relationship between  $\pi$  and  $P^\perp$  in  $\text{PG}(6, q)$ .

## 2 The Bruck-Bose representation of $\text{PG}(2, q^3)$ in $\text{PG}(6, q)$

### 2.1 The Bruck-Bose representation

We begin by describing the Bruck-Bose representation of  $\text{PG}(2, q^3)$  in  $\text{PG}(6, q)$ , and introduce the notation we will use.

A 2-*spread* of  $\text{PG}(5, q)$  is a set of  $q^3 + 1$  planes that partition  $\text{PG}(5, q)$ . A 2-*regulus* of  $\text{PG}(5, q)$  is a set of  $q + 1$  mutually disjoint planes  $\pi_1, \dots, \pi_{q+1}$  with the property that if a line meets three of the planes, then it meets all  $q + 1$  of them. Three mutually disjoint planes in  $\text{PG}(5, q)$  lie on a unique 2-regulus. A 2-spread  $\mathcal{S}$  is *regular* if for any three planes in  $\mathcal{S}$ , the 2-regulus containing them is contained in  $\mathcal{S}$ . See [11] for more information on 2-spreads.

The following construction of a regular 2-spread of  $\text{PG}(5, q)$  will be needed. Embed  $\text{PG}(5, q)$  in  $\text{PG}(5, q^3)$  and let  $g$  be a line of  $\text{PG}(5, q^3)$  disjoint from  $\text{PG}(5, q)$ . Let  $g^q, g^{q^2}$  be the conjugate lines of  $g$ ; both of these are disjoint from  $\text{PG}(5, q)$ . Let  $P_i$  be a point on  $g$ ; then the plane  $\langle P_i, P_i^q, P_i^{q^2} \rangle$  meets  $\text{PG}(5, q)$  in a plane. As  $P_i$  ranges over all the points of  $g$ , we get  $q^3 + 1$  planes of  $\text{PG}(5, q)$  that partition  $\text{PG}(5, q)$ . These planes form a regular spread  $\mathcal{S}$  of  $\text{PG}(5, q)$ . The lines  $g, g^q, g^{q^2}$  are called the (conjugate skew) *transversal lines* of the spread  $\mathcal{S}$ . Conversely, given a regular 2-spread in  $\text{PG}(5, q)$ , there is a unique set of three (conjugate skew) transversal lines in  $\text{PG}(5, q^3)$  that generate  $\mathcal{S}$  in this way.

We will use the linear representation of a finite translation plane  $\mathcal{P}$  of dimension at most three over its kernel, due independently to André [1] and Bruck and Bose [6, 7]. Let  $\Sigma_\infty$  be a hyperplane of  $\text{PG}(6, q)$  and let  $\mathcal{S}$  be a 2-spread of  $\Sigma_\infty$ . We use the phrase *a subspace of  $\text{PG}(6, q) \setminus \Sigma_\infty$*  to mean a subspace of  $\text{PG}(6, q)$  that is not contained in  $\Sigma_\infty$ . Consider the following incidence structure: the *points* of  $\mathcal{A}(\mathcal{S})$  are the points of  $\text{PG}(6, q) \setminus \Sigma_\infty$ ; the *lines* of  $\mathcal{A}(\mathcal{S})$  are the 3-spaces of  $\text{PG}(6, q) \setminus \Sigma_\infty$  that contain an element of  $\mathcal{S}$ ; and *incidence* in  $\mathcal{A}(\mathcal{S})$  is induced by incidence in  $\text{PG}(6, q)$ . Then the incidence structure  $\mathcal{A}(\mathcal{S})$  is an affine plane of order  $q^3$ . We can complete  $\mathcal{A}(\mathcal{S})$  to a projective plane  $\mathcal{P}(\mathcal{S})$ ; the points on the line at infinity  $\ell_\infty$  have a natural correspondence to the elements of the 2-spread  $\mathcal{S}$ . The projective plane  $\mathcal{P}(\mathcal{S})$  is the Desarguesian plane  $\text{PG}(2, q^3)$  if and only if  $\mathcal{S}$  is a regular 2-spread of  $\Sigma_\infty \cong \text{PG}(5, q)$  (see [5]).

We use the following notation: if  $P$  is an affine point of  $\text{PG}(3, q^3)$ , we also use  $P$  to refer to the corresponding affine point in  $\text{PG}(6, q)$ . If  $T$  is a point of  $\ell_\infty$  in  $\text{PG}(2, q^3)$ , we use  $[T]$  to refer to the spread element of  $\mathcal{S}$  in  $\text{PG}(6, q)$  corresponding to  $T$ . More generally, if  $X$  is a set of points of  $\text{PG}(2, q^3)$ , then we let  $[X]$  denote the corresponding set in  $\text{PG}(6, q)$ .

In the case  $\mathcal{P}(\mathcal{S}) \cong \text{PG}(2, q^3)$ , we can relate the coordinates of  $\text{PG}(2, q^3)$  and  $\text{PG}(6, q)$  as follows. Let  $\tau$  be a primitive element in  $\text{GF}(q^3)$  with primitive polynomial

$$x^3 - t_2x^2 - t_1x - t_0.$$

Then every element  $\alpha \in \text{GF}(q^3)$  can be uniquely written as  $\alpha = a_0 + a_1\tau + a_2\tau^2$  with  $a_0, a_1, a_2 \in \text{GF}(q)$ . Points in  $\text{PG}(2, q^3)$  have homogeneous coordinates  $(x, y, z)$  with  $x, y, z \in \text{GF}(q^3)$ . Let the line at infinity  $\ell_\infty$  have equation  $z = 0$ ; so the affine points of  $\text{PG}(2, q^3)$  have coordinates  $(x, y, 1)$ . Points in  $\text{PG}(6, q)$  have homogeneous coordinates  $(x_0, x_1, x_2, y_0, y_1, y_2, z)$  with  $x_0, x_1, x_2, y_0, y_1, y_2, z \in \text{GF}(q)$ . Let  $\Sigma_\infty$  have equation  $z = 0$ . Let  $P = (\alpha, \beta, 1)$  be a point of  $\text{PG}(2, q^3)$ . We can write  $\alpha = a_0 + a_1\tau + a_2\tau^2$  and  $\beta = b_0 + b_1\tau + b_2\tau^2$  with  $a_0, a_1, a_2, b_0, b_1, b_2 \in \text{GF}(q)$ . Then the map

$$\begin{aligned} \sigma: \text{PG}(2, q^3) \setminus \ell_\infty &\rightarrow \text{PG}(6, q) \setminus \Sigma_\infty \\ (\alpha, \beta, 1) &\mapsto (a_0, a_1, a_2, b_0, b_1, b_2, 1) \end{aligned}$$

is the Bruck-Bose map. More generally, if  $z \in \text{GF}(q)$ , then we can generalise this to

$$\sigma(\alpha, \beta, z) = (a_0, a_1, a_2, b_0, b_1, b_2, z).$$

Note that if  $z = 0$ , then  $T = (\alpha, \beta, 0)$  is a point of  $\ell_\infty$ , and  $\sigma(\alpha, \beta, 0)$  is a single point in the spread element  $[T]$  corresponding to  $T$ .

## 2.2 Subplanes and sublines in the Bruck-Bose representation

An *order- $q$ -subline* of  $\text{PG}(1, q^3)$  is defined to be one of the images of  $\text{PG}(1, q) = \{(a, 1) \mid a \in \text{GF}(q)\} \cup \{(1, 0)\}$  under  $\text{PGL}(2, q^3)$ . An *order- $q$ -subplane* of  $\text{PG}(2, q^3)$  is a subplane of  $\text{PG}(2, q^3)$  of order  $q$ . Equivalently, it is an image of  $\text{PG}(2, q)$  under  $\text{PGL}(3, q^3)$ . An *order- $q$ -subline* of  $\text{PG}(2, q^3)$  is a line of an order- $q$ -subplane of  $\text{PG}(2, q^3)$ .

In [2, 3], the authors determine the representation of order- $q$ -subplanes and order- $q$ -sublines of  $\text{PG}(2, q^3)$  in the Bruck-Bose representation in  $\text{PG}(6, q)$ , and we quote the results we need here. We first introduce some notation to simplify the statements. A *special conic*  $\mathcal{C}$  is a non-degenerate conic in a spread element (a plane), such that when we extend  $\mathcal{C}$  to  $\text{PG}(6, q^3)$ , it meets the transversals of the regular spread  $\mathcal{S}$ . Similarly, a *special twisted cubic*  $\mathcal{N}$  is a twisted cubic in a 3-space of  $\text{PG}(6, q) \setminus \Sigma_\infty$  about a spread element, such that when we extend  $\mathcal{N}$  to  $\text{PG}(6, q^3)$ , it meets the transversals of  $\mathcal{S}$ . Note that a special twisted cubic has no points in  $\Sigma_\infty$ .

**Theorem 2.1** [2] *Let  $b$  be an order- $q$ -subline of  $\text{PG}(2, q^3)$ .*

1. *If  $b \subset \ell_\infty$ , then in  $\text{PG}(6, q)$ ,  $b$  corresponds to a 2-regulus of  $\mathcal{S}$ . Conversely every 2-regulus of  $\mathcal{S}$  corresponds to an order- $q$ -subline of  $\ell_\infty$ .*
2. *If  $b$  meets  $\ell_\infty$  in a point, then in  $\text{PG}(6, q)$ ,  $b$  corresponds to a line of  $\text{PG}(6, q) \setminus \Sigma_\infty$ . Conversely every line of  $\text{PG}(6, q) \setminus \Sigma_\infty$  corresponds to an order- $q$ -subline of  $\text{PG}(2, q^3)$  tangent to  $\ell_\infty$ .*
3. *If  $b$  is disjoint from  $\ell_\infty$ , then in  $\text{PG}(6, q)$ ,  $b$  corresponds to a special twisted cubic. Further, a twisted cubic  $\mathcal{N}$  of  $\text{PG}(6, q)$  corresponds to an order- $q$ -subline of  $\text{PG}(2, q^3)$  if and only if  $\mathcal{N}$  is special.*

**Theorem 2.2** [2] *Let  $\mathcal{B}$  be an order- $q$ -subplane of  $\text{PG}(2, q^3)$ .*

1. *If  $\mathcal{B}$  is secant to  $\ell_\infty$ , then in  $\text{PG}(6, q)$ ,  $\mathcal{B}$  corresponds to a plane of  $\text{PG}(6, q)$  that meets  $q + 1$  spread elements. Conversely, any plane of  $\text{PG}(6, q)$  that meets  $q + 1$  spread elements corresponds to an order- $q$ -subplane of  $\text{PG}(2, q^3)$  secant to  $\ell_\infty$ .*
2. *Suppose  $\mathcal{B}$  is tangent to  $\ell_\infty$  in the point  $T$ . Then  $\mathcal{B}$  determines a set  $[\mathcal{B}]$  of points in  $\text{PG}(6, q)$  (where the affine points of  $\mathcal{B}$  correspond to the affine points of  $[\mathcal{B}]$ ) such that:*
  - (a)  *$[\mathcal{B}]$  is a ruled surface with conic directrix  $\mathcal{C}$  contained in the plane  $[T] \in \mathcal{S}$ , and twisted cubic directrix  $\mathcal{N}$  contained in a 3-space  $\Sigma$  that meets  $\Sigma_\infty$  in a spread element (distinct from  $[T]$ ). The points of  $[\mathcal{B}]$  lie on  $q + 1$  pairwise disjoint generator lines joining  $\mathcal{C}$  to  $\mathcal{N}$ .*
  - (b) *The  $q + 1$  generator lines of  $[\mathcal{B}]$  joining  $\mathcal{C}$  to  $\mathcal{N}$  are determined by a projectivity from  $\mathcal{C}$  to  $\mathcal{N}$ .*

- (c) When we extend  $[\mathcal{B}]$  to  $\text{PG}(6, q^3)$ , it contains the conjugate transversal lines  $g, g^q, g^{q^2}$  of the regular spread  $\mathcal{S}$ . So  $\mathcal{C}$  and  $\mathcal{N}$  are special.
- (d)  $[\mathcal{B}]$  is the intersection of nine quadrics in  $\text{PG}(6, q)$ .

## 2.3 The collineation group in the Bruck-Bose representation

Consider a homography  $\alpha \in \text{PGL}(3, q^3)$ , acting on  $\text{PG}(2, q^3)$ , which fixes  $\ell_\infty$  as a set of points. There is a corresponding homography  $[\alpha] \in \text{PGL}(7, q)$  acting on the Bruck Bose representation of  $\text{PG}(2, q^3)$  as  $\text{PG}(6, q)$ . Note that  $[\alpha]$  fixes the hyperplane  $\Sigma_\infty$  at infinity of  $\text{PG}(6, q)$ , and permutes the elements of the regular spread  $\mathcal{S}$  in  $\Sigma_\infty$ . Consequently subgroups of  $\text{PGL}(3, q^3)$  fixing  $\ell_\infty$  correspond to subgroups of  $\text{PGL}(7, q)$  fixing  $\Sigma_\infty$  and permuting the spread elements in  $\Sigma_\infty$ . For more details, see [4]. Hence when we prove results about transitivity in  $\text{PG}(2, q^3)$ , there is a corresponding transitivity result in  $\text{PG}(6, q)$ . We are interested in a particular Singer cycle acting on  $\text{PG}(6, q)$ . It is straightforward to prove the following.

**Theorem 2.3** *Consider the homography  $\Theta \in \text{PGL}(7, q)$  with  $7 \times 7$  matrix*

$$M = \begin{pmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where } T = \begin{pmatrix} 0 & 0 & t_0 \\ 1 & 0 & t_1 \\ 0 & 1 & t_2 \end{pmatrix}.$$

*Then in  $\text{PG}(6, q)$ ,  $\Theta$  fixes each plane of the spread  $\mathcal{S}$ , and  $\langle \Theta \rangle$  acts regularly on the set of points, and on the set of lines, of each spread element.*

## 3 Sublines contained in a tangent splash

Let  $\pi$  be an order- $q$ -subplane of  $\text{PG}(2, q^3)$  tangent to  $\ell_\infty$  at the point  $T$ . Recall that the *tangent splash*  $\mathcal{S}_T$  of  $\pi$  is the set of points of  $\ell_\infty$  lying on lines of  $\pi$ , and  $T = \pi \cap \ell_\infty$  is called the *centre* of  $\mathcal{S}_T$ . As two distinct lines of  $\pi$  meet in a point of  $\pi$ , the lines of  $\pi$  not through  $T$  meet  $\ell_\infty$  in distinct points. Hence a tangent splash has  $q^2 + 1$  points. We say a set of  $q^2 + 1$  points  $\mathcal{S}_T \subset \ell_\infty$  is a tangent splash with centre  $T$  if there is an order- $q$ -subplane  $\pi$  with tangent splash  $\mathcal{S}_T$  and  $\pi \cap \ell_\infty = T$ .

In this section we investigate the order- $q$ -sublines contained in a tangent splash  $\mathcal{S}_T$  and show that the points of a tangent splash form an affine plane. The next result investigates order- $q$ -sublines contained in  $\mathcal{S}_T$  which contain the centre  $T$ . We show later in Theorem 7.1 that every order- $q$ -subline contained in  $\mathcal{S}_T$  must contain the centre  $T$ .

**Lemma 3.1** *Let  $\pi$  be an order- $q$ -subplane of  $\text{PG}(2, q^3)$  tangent to  $\ell_\infty$  at the point  $T$  with tangent splash  $\mathcal{S}_T$ .*

1. The lines of  $\pi$  through an affine point  $P \in \pi$  meet  $\mathcal{S}_T$  in an order- $q$ -subline  $t_P$ . Further, there is a bijection between the  $q^2 + q$  affine points  $P$  of  $\pi$  and the order- $q$ -sublines  $t_P$  containing  $T$  and contained in  $\mathcal{S}_T$ .
2. Let  $U, V$  be distinct points non-centre points of  $\mathcal{S}_T$ . Then the unique order- $q$ -subline containing  $T, U, V$  is contained in  $\mathcal{S}_T$ .

**Proof** Label the lines of  $\pi$  through  $P$  by  $\ell_0 = PT, \ell_1, \dots, \ell_q$ . For each  $i$ , let  $\bar{\ell}_i$  denote the extension of  $\ell_i$  to  $\text{PG}(2, q^3)$ . If  $m$  is another line of  $\pi$ ,  $P \notin m$ , then the points  $\ell_0 \cap m, \dots, \ell_q \cap m$  are the  $q + 1$  distinct points of  $m$ . Hence the points  $t_P = \{T = \bar{\ell}_0 \cap \ell_\infty, \bar{\ell}_1 \cap \ell_\infty, \dots, \bar{\ell}_q \cap \ell_\infty\}$  are the projection from  $P$  of an order- $q$ -subline  $m$  onto the line  $\ell_\infty$ , and so  $t_P$  is an order- $q$ -subline of  $\ell_\infty$ . As the order- $q$ -sublines  $\ell_i$  are lines of  $\pi$ , the points  $\bar{\ell}_i \cap \ell_\infty$  are points of  $\mathcal{S}_T$ , so  $t_P$  is contained in  $\mathcal{S}_T$  and contains the centre  $T$ .

As two lines of  $\pi$  meet in  $\pi$ , through a non-centre point  $U$  of  $\mathcal{S}_T$  there is unique line  $u$  of  $\pi$ . So an order- $q$ -subline of  $\mathcal{S}_T$  containing  $T$  is projected by at most one point of  $\pi$ . Hence for distinct affine points  $P, Q$  of  $\pi$ , the order- $q$ -sublines  $t_P$  and  $t_Q$  are distinct. There are  $q^2 + q$  points of  $\pi$  distinct from  $T$ . We show in the next paragraph that there are  $q^2 + q$  order- $q$ -sublines contained in  $\mathcal{S}_T$  and containing  $T$ , and hence we have a bijection from affine points  $P \in \pi$  to order- $q$ -sublines  $t_P$  contained in  $\mathcal{S}_T$  and containing  $T$ .

Let  $U, V$  be distinct points of  $\mathcal{S}_T$ ,  $U, V \neq T$ , and let  $u, v$  be the distinct lines in  $\pi$  meeting  $\ell_\infty$  in  $U, V$  respectively. Let  $P = u \cap v$ , so  $P \in \pi$  and  $P \neq T$  as  $T \notin u, v$ . The lines of  $\pi$  through  $P$  meet  $\ell_\infty$  in an order- $q$ -subline  $t_P$  containing  $T, U, V$ . Hence  $t_P$  is the unique order- $q$ -subline through  $T, U, V$ . By part 1,  $t_P$  is contained in  $\mathcal{S}_T$ , completing the proof of part 2. As there are  $\binom{q^2}{2} / \binom{q}{2}$  ways to choose  $U, V$  to get a unique order- $q$ -subline,  $\mathcal{S}_T$  contains exactly  $q(q + 1)$  order- $q$ -sublines containing  $T$ , completing the proof of part 1.

□

Lemma 3.1 has a natural correspondence in the Bruck-Bose representation in  $\text{PG}(6, q)$ . For example, part 2 says that given a tangent splash  $[\mathcal{S}_T]$  which consists of  $q^2 + 1$  spread elements, the unique 2-regulus containing the centre  $[T]$  and two further elements  $[U], [V]$  of  $[\mathcal{S}_T]$  is contained in  $[\mathcal{S}_T]$ .

The next result shows how to construct an affine plane from the points and order- $q$ -sublines of a tangent splash.

**Theorem 3.2** *Consider the incidence structure  $\mathcal{I}$  with points the  $q^2$  elements of  $\mathcal{S}_T \setminus \{T\}$ , lines the order- $q$ -sublines containing  $T$  and contained in  $\mathcal{S}_T$ , and incidence is inclusion. Then  $\mathcal{I}$  is an affine plane of order  $q$ .*

**Proof** By Lemma 3.1(2), two points of  $\mathcal{I}$  lie in a unique line of  $\mathcal{I}$ . Hence  $\mathcal{I}$  is a  $2-(q^2, q, 1)$  design and so is an affine plane of order  $q$ . □

## 4 Group properties of tangent subplanes and tangent splashes

In this section we study the collineation group of  $\text{PG}(2, q^3)$  fixing  $\ell_\infty$  and show it is transitive on order- $q$ -subplanes tangent to  $\ell_\infty$ , and hence transitive on tangent splashes on  $\ell_\infty$ . Thus all tangent subplanes have the same geometrical and group properties, and to prove results about tangent subplanes in general, we may coordinatise a particular tangent subplane, and work with that. The two main results of this section are:

**Theorem 4.1** *The subgroup of  $\text{PGL}(3, q^3)$  acting on  $\text{PG}(2, q^3)$  and fixing the line  $\ell_\infty$  is transitive on all the order- $q$ -subplanes meeting  $\ell_\infty$  in a point.*

**Corollary 4.2** *The subgroup of  $\text{PGL}(3, q^3)$  acting on  $\text{PG}(2, q^3)$  and fixing the line  $\ell_\infty$  is transitive on the tangent splashes of  $\ell_\infty$ .*

We begin by counting order- $q$ -subplanes in  $\text{PG}(2, q^3)$ .

**Lemma 4.3** *In  $\text{PG}(2, q^3)$  there are  $q^6(q^6 + q^3 + 1)(q^2 - q + 1)(q^2 + q + 1)$  order- $q$ -subplanes in total, and  $q^7(q^3 - 1)(q^3 + 1)(q^2 + q + 1)$  order- $q$ -subplanes tangent to a given line.*

**Proof** It is straightforward to count the number of order- $q$ -subplanes in  $\text{PG}(2, q^3)$  as each quadrangle of  $\text{PG}(2, q^3)$  is contained in a unique order- $q$ -subplane. To count  $x$ , the number of order- $q$ -subplanes tangent to a fixed line, we count in two ways the number of pairs  $(\ell, \pi)$  where  $\ell$  is a line of  $\text{PG}(2, q^3)$  tangent to an order- $q$ -subplane  $\pi$ . This gives

$$\begin{aligned} x &= q^6(q^6 + q^3 + 1)(q^2 - q + 1)(q^2 + q + 1) \times (q^2 + q + 1)(q^3 - q)/(q^6 + q^3 + 1) \\ &= q^7(q^3 - 1)(q^3 + 1)(q^2 + q + 1). \end{aligned}$$

□

The following lemma provides the proof for Theorem 4.1 and Corollary 4.2, as well as some other collineation results that are used in later sections.

**Lemma 4.4** *Consider the collineation group  $G = \text{PGL}(3, q^3)$  acting on  $\text{PG}(2, q^3)$ . Let  $K = G_\pi$  be the subgroup of  $G$  fixing an order- $q$ -subplane  $\pi$ ; let  $H = G_\ell$  be the subgroup of  $G$  fixing a line  $\ell$  of  $\text{PG}(2, q^3)$ ; and let  $I = G_{\pi, \ell}$  be the subgroup of  $G$  fixing a line  $\ell$  and an order- $q$ -subplane  $\pi$  tangent to  $\ell$ . Then*

1.  $G = \text{PGL}(3, q^3)$  is transitive on the order- $q$ -subplanes of  $\text{PG}(2, q^3)$ .

2.  $K = G_\pi$  has three orbits on the points of  $\text{PG}(2, q^3)$ : the points of  $\pi$ , the points of  $\text{PG}(2, q^3) \setminus \pi$  that lie on a line of  $\pi$ , and the points that do not lie on a line of  $\pi$ .  $K$  has three line orbits: lines of  $\pi$ , lines of  $\text{PG}(2, q^3)$  tangent to  $\pi$ , and lines of  $\text{PG}(2, q^3)$  exterior to  $\pi$ .
3.  $|H| = q^9(q^3 - 1)^2(q^3 + 1)$ .
4.  $|I| = q^2(q - 1)$ .
5.  $H = G_\ell$  is transitive on all the order- $q$ -subplanes tangent to  $\ell$ .
6.  $I = G_{\pi, \ell}$  fixes linewise the lines of  $\pi$  through  $T = \pi \cap \ell$ , and is transitive on the points of  $\pi \setminus \{T\}$  on these lines. Further  $I$  is transitive on the lines of  $\pi$  not through  $T$ .
7.  $I = G_{\pi, \ell_\infty}$  fixes the tangent splash of  $\pi$  and is transitive on the non-centre points of this splash.

**Proof** Part 1 is an immediate corollary of the Fundamental Theorem of Projective Geometry (see [10]). Hence when we consider  $K = G_\pi$ , we can choose  $\pi = \text{PG}(2, q)$ . Now  $K$  fixes  $\pi = \text{PG}(2, q)$  as a set and acts faithfully on  $\text{PG}(2, q)$  (as the only collineations of  $\text{PG}(2, q^3)$  that fix  $\pi$  pointwise are the automorphic collineations in  $\text{P}\Gamma\text{L}(3, q^3) \setminus \text{PGL}(3, q^3)$ ). Hence  $K = \text{PGL}(3, q)$ , and so  $K$  is transitive on the points and lines of  $\pi$ . Note that  $|K| = |\text{PGL}(3, q)| = q^3(q^3 - 1)(q^2 - 1)$ , see for example [9]. Hence the points of  $K$  form one point orbit of  $K$ . Consider the line  $[1, -1, 0]$  which meets  $\pi = \text{PG}(2, q)$  in a line of  $\pi$ . The point  $P = (1, 1, \tau)$  is on the line  $[1, -1, 0]$  but not in  $\pi$ . We find the subgroup  $K_P$  of  $K$  fixing  $P$ . Suppose

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \tau \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 1 \\ \tau \end{pmatrix}$$

(where  $a, \dots, i \in \text{GF}(q)$ ). This gives  $a + b + c\tau = d + e + f\tau$  (and so  $c = f$ ,  $a + b = d + e$ ) and  $(a + b + c\tau)\tau = g + h + i\tau$  (and so  $c = 0$ ,  $a + b = i$ ,  $g + h = 0$ ). Hence the matrix is

$$\begin{pmatrix} a & b & 0 \\ d & a + b - d & 0 \\ g & -g & a + b \end{pmatrix}.$$

As the determinant is non-zero, we may take  $a + b = 1$  and the matrix is

$$\begin{pmatrix} a & 1 - a & 0 \\ d & 1 - d & 0 \\ g & -g & 1 \end{pmatrix}$$

which has non-zero determinant if and only if  $a \neq d$ . Hence the size of  $K_P$  is  $q^2(q - 1)$ . The orbit stabilizer theorem gives  $|P^K| = |K|/|K_P| = (q^3(q^3 - 1)(q^2 - 1))/(q^2(q - 1)) = (q^2 + q + 1)(q^3 - q)$ . This is equal to the number of points of  $\text{PG}(2, q^3) \setminus \pi$  that lie on a line of  $\pi$ . Hence the points of  $\text{PG}(2, q^3) \setminus \pi$  that lie on a line of  $\pi$  form an orbit of  $K$ .



We now consider the points of  $\text{PG}(2, q^3) \setminus \pi$  that do not lie on a line of  $\pi$ . The point  $Q = (1, \tau, \tau^2)$  does not lie on any lines of  $\pi = \text{PG}(2, q)$ . We find the subgroup  $K_Q$  of  $K$  fixing  $Q$ . Suppose the matrix

$$B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

(where  $a, \dots, i \in \text{GF}(q)$ ) fixes the point  $Q$ . Then  $(a + b\tau + c\tau^2)\tau = d + e\tau + f\tau^2$  and  $(d + e\tau + f\tau^2)\tau = g + h\tau + i\tau^2$ , so we have

$$B = \begin{pmatrix} a & b & c \\ ct_0 & a + ct_1 & b + ct_2 \\ t_0(b + ct_2) & ct_0 + (b + ct_2)t_1 & a + ct_1 + (b + ct_2)t_2 \end{pmatrix} = \begin{pmatrix} \mathbf{x}^t \\ (T\mathbf{x})^t \\ (T^2\mathbf{x})^t \end{pmatrix}$$

where  $\mathbf{x}^t = (a, b, c)$  and  $T$  is the matrix given in Theorem 2.3. The matrix  $T$  has order  $q^2 + q + 1$ , and generates a Singer cycle on the points of  $\pi = \text{PG}(2, q)$ . Hence the points  $\mathbf{x}, T\mathbf{x}, T^2\mathbf{x}$  of  $\pi$  are not collinear, and so the matrix  $B$  has nonzero determinant provided  $a, b, c$  are not all zero. Further  $(a, b, c)$  and  $y(a, b, c)$  give the same collineation if and only if  $y \in \text{GF}(q) \setminus \{0\}$ . Thus the subgroup  $K_Q$  is of size  $(q^3 - 1)/(q - 1) = q^2 + q + 1$ . Note that setting  $(a, b, c) = (0, 1, 0)$  gives the matrix  $T^t$  which fixes  $Q$ , so as the matrix  $T$  generates a Singer cycle of order  $q^2 + q + 1$ , the subgroup  $K_Q$  is the Singer cycle generated by the matrix  $T^t$ . By the orbit stabilizer theorem,  $|Q^K| = |K|/|K_Q| = (q^3(q^3 - 1)(q^2 - 1))/(q^2 + q + 1) = q^3(q - 1)(q^2 - 1)$ . This is equal to the number of points of  $\text{PG}(2, q^3) \setminus \pi$  that lie on no line of  $\pi$ , hence these points form an orbit of  $K$ . The lines orbits of  $K$  follow by a dual argument, completing the proof of part 2.

As  $G$  is transitive on the  $q^6 + q^3 + 1$  lines of  $\text{PG}(2, q^3)$ , the orbit stabilizer theorem gives  $|H| = |G|/(q^6 + q^3 + 1) = q^9(q^3 - 1)^2(q^3 + 1)(q^6 + q^3 + 1)/(q^6 + q^3 + 1) = q^9(q^3 - 1)^2(q^3 + 1)$ , proving part 3.

Consider the subgroup  $I$  fixing a order- $q$ -subplane  $\pi$  and a line  $\ell$  tangent to  $\pi$ . By part 1 we can choose  $\pi = \text{PG}(2, q)$  and so  $G_\pi = \text{PGL}(3, q)$ . By part 2, without loss of generality we can pick  $\ell = [0, 1, \tau] = \{(1, 0, 0)\} \cup \{(x, -\tau, 1) \mid x \in \text{GF}(q^3)\}$ . Note that  $\ell$  is tangent to  $\pi = \text{PG}(2, q)$  in the point  $T = (1, 0, 0)$ . If an element of  $\text{PGL}(3, q)$  fixes  $T = (1, 0, 0)$  and fixes  $\ell$ , then it has form

$$A = \begin{pmatrix} a & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b, c \in \text{GF}(q), \quad a \neq 0. \quad (1)$$

Thus  $I = G_{\pi, \ell}$  consists of matrices of this form, and so  $|I| = q^2(q - 1)$ , proving part 4. Note that only the identity element of  $I$  fixes  $\ell$  pointwise.

For part 5, the orbit stabilizer theorem gives  $|\pi^H| = |H|/|I|$ . Using parts 3 and 4 gives  $|\pi^H| = q^7(q^3 - 1)(q^3 + 1)(q^2 + q + 1)$ . By Lemma 4.3, this is the number of order- $q$ -subplanes tangent to  $\ell$ , so  $H$  is transitive on order- $q$ -subplanes tangent to  $\ell$ . Note that this completes the proof of Theorem 4.1. As  $H$  is transitive on order- $q$ -subplanes tangent to  $\ell_\infty$ , it is transitive on tangent splashes of  $\ell_\infty$ , proving Corollary 4.2.

For part 6, as before we can without loss of generality let  $\pi = \text{PG}(2, q)$ ,  $\ell = [0, 1, \tau]$  and consider  $I = G_{\pi, \ell}$ . We know that the matrices of  $I$  have form given in equation (1). So a point  $P = (x, y, z) \in \pi$  ( $x, y, z \in \text{GF}(q)$ ) is mapped under an element of  $I$  to  $P' = (ax + by + cz, y, z)$ , which lies on the line  $PT$ . Hence the line  $PT$  is fixed by  $I$ . Further, the points of the order- $q$ -subline  $PT \cap \pi$  distinct from  $T$  have coordinates  $(\lambda + x, y, z)$ , for some  $\lambda \in \text{GF}(q)$ , so we can choose  $a, b, c \in \text{GF}(q)$  so that  $P'$  is any point of the line  $PT$ . Hence  $I$  is transitive on the points of  $(PT \cap \pi) \setminus \{T\}$ .

Next let  $m = [r, s, t]$ ,  $r, s, t \in \text{GF}(q)$  be any line  $\pi$  not through  $T$ , so  $r \neq 0$ . Using the general form of an element of  $I$  from equation (1), the line  $m$  is mapped to the line  $[r, -br + as, -cr + at]$  for some  $a, b, c \in \text{GF}(q)$ ,  $a \neq 0$ . This line is equal to  $m$  if and only if  $b = s(a - 1)/r$  and  $c = t(a - 1)/r$ . Thus the subgroup of  $I$  fixing a line  $m$  of  $\pi$  not through  $T$  is of size  $q - 1$  (one for each possible value of  $a$ ). Hence by the orbit stabilizer theorem, the orbit of  $m$  in  $I$  is of size  $q^2(q - 1)/(q - 1) = q^2$ , which is equal to the number of lines of  $\pi$  not through  $T$ . Hence  $I$  is transitive on lines of  $\pi$  not through  $T$ , proving part 6.

For part 7, let  $I$  be the subgroup of  $G$  fixing  $\ell_\infty$  and an order- $q$ -subplane  $\pi$  tangent to  $\ell_\infty$ . As  $I$  is transitive on the lines of  $\pi$  not through  $T = \pi \cap \ell_\infty$  (by part 6),  $I$  fixes the tangent splash  $\mathcal{S}_T$  of  $\pi$  on  $\ell_\infty$  and is transitive on the non-centre points of  $\mathcal{S}_T$ .  $\square$

## 5 Counting tangent splashes

In this section we count the number of tangent splashes on  $\ell_\infty$ . Further we investigate the number of points needed to determine a unique order- $q$ -subplane with a given tangent splash.

**Theorem 5.1** *Let  $\mathcal{S}_T$  be a tangent splash of  $\ell_\infty$  and let  $\ell$  be an order- $q$ -subline disjoint from  $\ell_\infty$  lying on a line of  $\text{PG}(2, q^3)$  which meets  $\mathcal{S}_T \setminus \{T\}$ . Then there is a unique tangent order- $q$ -subplane that contains  $\ell$  and has tangent splash  $\mathcal{S}_T$ .*

**Proof** We prove this result using coordinates. Everywhere else in this article we let the line at infinity  $\ell_\infty$  of  $\text{PG}(2, q^3)$  have homogeneous coordinates  $[0, 0, 1]$ . Here we give the line at infinity the coordinates  $[0, 1, \tau]$ , and to avoid confusion, denote it by  $\ell'_\infty$ . Consider the order- $q$ -subplane  $\pi = \text{PG}(2, q)$ , it is tangent to  $\ell'_\infty$  at the point  $T = (1, 0, 0)$ . The lines of  $\pi$  not through  $T$  have coordinates  $[1, r, s]$ ,  $r, s \in \text{GF}(q)$ , and meet  $\ell'_\infty$  in the points  $(s - r\tau, \tau, -1)$ . So the tangent splash of  $\pi$  onto  $\ell'_\infty$  is  $\mathcal{S}_T = \{T = (1, 0, 0)\} \cup \{(s - r\tau, \tau, -1) \mid r, s \in \text{GF}(q)\}$ . Note that the set  $\ell = \{(0, c, 1) \mid c \in \text{GF}(q) \cup \{\infty\}\}$  is an order- $q$ -subline of  $\pi$  that is disjoint from  $\ell'_\infty$ .

By Corollary 4.2 and Lemma 4.4(6), without loss of generality we only need prove the theorem holds for the splash  $\mathcal{S}_T = \{T = (1, 0, 0)\} \cup \{(s - r\tau, \tau, -1) \mid r, s \in \text{GF}(q)\}$  and

the order- $q$ -subline  $\ell = \{(0, c, 1) \mid c \in \text{GF}(q) \cup \{\infty\}\}$ . It suffices to show that the only order- $q$ -subplane containing  $\ell$  with tangent splash  $\mathcal{S}_T$  is  $\pi = \text{PG}(2, q)$ .

Any point in an order- $q$ -subplane containing  $T$  and  $\ell$  will be on a line joining  $T$  to a point of  $\ell$ . Such a point has coordinates  $(\alpha, 1, 0)$  or  $(\alpha, c, 1)$  for  $\alpha \in \text{GF}(q^3)$ ,  $c \in \text{GF}(q)$ . Consider one such point  $X = (\alpha, d, 1)$  for some  $\alpha \in \text{GF}(q^3)$ ,  $d \in \text{GF}(q)$ . The set of points  $T, X, \ell$  contains a quadrangle and so lie in a unique order- $q$ -subplane  $\pi'$ . We show that  $\pi'$  has splash  $\mathcal{S}_T = \{T = (1, 0, 0)\} \cup \{(s - r\tau, \tau, -1) \mid r, s \in \text{GF}(q)\}$  if and only if  $\alpha \in \text{GF}(q)$ , and so  $\pi' = \text{PG}(2, q)$ .

Let  $m$  be the unique order- $q$ -subline determined by  $T = (1, 0, 0)$ ,  $X = (\alpha, d, 1)$ , and  $(0, d, 1)$ . The homography with matrix

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & d & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

maps the order- $q$ -subline  $\{(a, 1, 0) \mid a \in \text{GF}(q) \cup \{\infty\}\}$  onto the order- $q$ -subline  $m$ , so  $m$  has points  $X_a = (a\alpha, d, 1)$  where  $a \in \text{GF}(q) \cup \{\infty\}$ . The line joining  $X_a$  to the point  $(0, c, 1)$  of  $\ell$  meets the line  $\ell'_\infty$  in the point  $(-a\alpha\tau - a\alpha c, -\tau(d - c), d - c)$ . As  $a, c, d \in \text{GF}(q)$ , this point belongs to the tangent splash  $\mathcal{S}_T$  only if  $\alpha\tau + \alpha c = s - r\tau$  for some  $r, s \in \text{GF}(q)$ . Writing  $\alpha = \alpha_0 + \alpha_1\tau + \alpha_2\tau^2$  for unique  $\alpha_0, \alpha_1, \alpha_2 \in \text{GF}(q)$ , we get

$$\alpha\tau + \alpha c = (\alpha_2t_0 + c\alpha_0) + (\alpha_0 + \alpha_2t_1 + c\alpha_1)\tau + (\alpha_1 + \alpha_2t_2 + c\alpha_2)\tau^2.$$

We want the coefficient of  $\tau^2$  to be zero for all  $c \in \text{GF}(q)$ , that is,  $\alpha_1 + \alpha_2t_2 + c\alpha_2 = 0$ . Setting  $c = -t_2$  gives  $\alpha_1 = 0$ , and setting  $c = -t_2 + 1$  gives  $\alpha_2 = 0$ . Hence  $\alpha = \alpha_0$  is in  $\text{GF}(q)$ , and so the point  $X_a$  lies in  $\text{PG}(2, q)$  for all  $a \in \text{GF}(q)$ . Hence the order- $q$ -subplane  $\pi'$  has splash  $\mathcal{S}_T$  if and only if  $\pi' = \text{PG}(2, q)$ . Hence there is a unique order- $q$ -subplane (namely  $\text{PG}(2, q)$ ) that contains  $T, \ell$ , and has tangent splash  $\mathcal{S}_T$ .  $\square$

In [4], it is shown how to geometrically construct the unique order- $q$ -subplane from a tangent splash of  $\ell_\infty$  and an order- $q$ -subline disjoint from  $\ell_\infty$ .

We now count the number of tangent splashes in  $\text{PG}(2, q^3)$ .

**Theorem 5.2** 1. *The number of tangent order- $q$ -subplanes with a given tangent splash is  $q^6(q^3 - 1)$ .*

2. *The number of tangent splashes on  $\ell_\infty$  is  $q(q^3 + 1)(q^2 + q + 1)$ .*

3. *The number of tangent splashes of  $\ell_\infty$  with a common centre is  $q(q^2 + q + 1)$ .*

**Proof** To prove part 1, let  $\mathcal{S}_T$  be a tangent splash on the line  $\ell_\infty$  in  $\text{PG}(2, q^3)$ . We count the pairs  $(\ell, \pi)$  where  $\pi$  is an order- $q$ -subplane tangent to  $\ell_\infty$  at  $T$  with splash  $\mathcal{S}_T$  and  $\ell$  is line of  $\pi$  not through  $T$ . First note that if  $m$  is a line of  $\text{PG}(2, q^3)$  and  $X$  a point of  $m$ ,

then it is straightforward to show that the number of order- $q$ -sublines of  $m$  that do not contain  $X$  is  $q^3(q^3 - 1)$ . There are  $q^2$  points in  $\mathcal{S}_T$  distinct from  $T$ , and each point lies on  $q^3$  lines of  $\text{PG}(2, q^3)$  distinct from  $\ell_\infty$ . Hence there are  $q^2 \times q^3 \times q^3(q^3 - 1)$  choices for an order- $q$ -subline  $\ell$ . By Theorem 5.1, each of these order- $q$ -sublines  $\ell$  lies on a unique order- $q$ -subplane that contains  $\ell$  and has tangent splash  $\mathcal{S}_T$ . Hence if  $n$  is the number of order- $q$ -subplanes with the given tangent splash, we have  $q^2 \times q^3 \times q^3(q^3 - 1) \times 1 = n \times q^2$  and so there are  $q^6(q^3 - 1)$  tangent order- $q$ -subplanes with a given tangent splash.

To count the number of tangent splashes on  $\ell_\infty$ , by Corollary 4.2 we can divide the total number of order- $q$ -subplanes tangent to  $\ell_\infty$  (calculated in Lemma 4.3) by the number of order- $q$ -subplanes with a given tangent splash (calculated in part 1). This proves part 2.

For part 3, the subgroup of  $\text{PGL}(3, q^3)$  fixing  $\ell_\infty$  is transitive on the points of  $\ell_\infty$ , hence each point of  $\ell_\infty$  is the centre of a constant number of splashes. The result now follows from part 2.  $\square$

Theorem 5.1 shows that a tangent splash and one affine order- $q$ -subline not through the centre uniquely determines an order- $q$ -subplane. We now consider the case of a tangent splash and one affine order- $q$ -subline through the centre, and show that this determines more than one order- $q$ -subplane.

**Theorem 5.3** *In  $\text{PG}(2, q^3)$ , let  $\mathcal{S}_T$  be a tangent splash of  $\ell_\infty$  with centre  $T$ . Let  $m$  be an order- $q$ -subline through  $T$  (not in  $\ell_\infty$ ). Then there are  $q(q^2 - 1)$  tangent order- $q$ -subplanes that contain  $m$  and have tangent splash  $\mathcal{S}_T$ .*

**Proof** First note that the subgroup of  $\text{PGL}(3, q^3)$  containing translations with axis  $\ell_\infty$  is 2-transitive on the affine points of an affine line. Hence the number of order- $q$ -subplanes containing two distinct affine points  $P, Q$  such that  $T \in PQ$  is a constant, denote it by  $x$ . As  $T, P, Q$  lie in a unique order- $q$ -subline,  $x$  is the number of order- $q$ -subplanes containing an affine order- $q$ -subline through  $T$ . We count in two ways the triples  $(P, Q, \pi)$  where  $P, Q$  are distinct affine points, the line  $PQ$  contains  $T$  and  $\pi$  is a tangent order- $q$ -subplane with splash  $\mathcal{S}_T$  containing  $P$  and  $Q$ . Using Lemma 5.2(1) we have  $q^6(q^3 - 1) \times x = q^6(q^3 - 1) \times (q^2 + q)(q - 1)$  and so  $x = q(q^2 - 1)$ .  $\square$

We now look at how many points of  $\ell_\infty$  are needed to uniquely determine a tangent splash.

**Theorem 5.4** *Two tangent splashes with a common centre  $T$  can meet in at most an order- $q$ -subline which contains  $T$ . Further, there are  $q + 1$  tangent splashes of  $\ell_\infty$  with centre  $T$  containing a fixed order- $q$ -subline through  $T$ .*

**Proof** Suppose two splashes  $\mathcal{S}_T$  and  $\mathcal{S}'_T$  have a common centre  $T$  and contain three further common points  $U, V, W$  not all on order- $q$ -subline through  $T$ . Denote by  $\ell(X, Y, Z)$  the unique order- $q$ -subline through any three distinct points  $X, Y, Z$  of  $\ell_\infty$ . By Lemma 3.1(2),  $\mathcal{S}_T$  and  $\mathcal{S}'_T$  both contain the order- $q$ -subline  $\ell(T, U, V) = \{T, U_1 = U, U_2, \dots, U_q\}$ . Further,

the  $q$  order- $q$ -sublines  $\ell(T, U_i, W)$ ,  $i = 1, \dots, q$ , are contained in both  $\mathcal{S}_T$  and  $\mathcal{S}'_T$ . These  $q$  order- $q$ -sublines together with  $\ell(T, U, V)$  cover  $(q+1) + q \times (q-1) = q^2 + 1$  points. Hence  $\mathcal{S}_T = \mathcal{S}'_T$ . Thus  $\mathcal{S}_T, \mathcal{S}'_T$  can both contain the order- $q$ -subline  $\ell(T, U, V)$ , but cannot have any further points in common.

Let  $x$  be the number of splashes with centre  $T$  containing a fixed order- $q$ -subline  $\ell$  of  $\ell_\infty$ , where  $\ell$  contains  $T$ . We count in two ways the pairs  $(\ell, \mathcal{S}_T)$  where  $\ell$  is an order- $q$ -subline of  $\ell_\infty$  through  $T$  and  $\mathcal{S}_T$  is a tangent splash with centre  $T$  containing  $\ell$ . By Lemma 3.1(1), the number of order- $q$ -sublines contained in  $\mathcal{S}_T$  and containing  $T$  is  $q^2 + q$ . So using Theorem 5.2(3), we have

$$q^2(q^2 + q + 1) \times x = q(q^2 + q + 1) \times q(q + 1),$$

so  $x = q + 1$ . That is, there are  $q + 1$  tangent splashes with common centre  $T$  and containing a common order- $q$ -subline  $\ell$  through  $T$ .  $\square$

**Theorem 5.5** *Let  $T, U, V, W$  be four distinct points of  $\ell_\infty$  not on a common order- $q$ -subline. Then there is a unique tangent splash containing  $T, U, V, W$  with centre  $T$ .*

**Proof** Let  $\ell$  be the unique order- $q$ -subline containing  $T, U, V$ . By Lemma 3.1(2), a tangent splash with centre  $T$  containing  $U, V$  must contain  $\ell$ . By Theorem 5.4, there are  $q + 1$  tangent splashes with centre  $T$  containing  $\ell$ . These tangent splashes cover  $(q^2 - q) \times (q + 1) + q + 1 = q^3 + 1$  points of  $\ell_\infty$ . Hence  $W$  lies in exactly one of these tangent splashes. Hence  $T$  and three further points of  $\ell_\infty$  not all on a common order- $q$ -subline lie in a unique tangent splash with centre  $T$ .  $\square$

## 6 Coordinatisation of a tangent subplane

We need to calculate the coordinates of points in an order- $q$ -subplane of  $\text{PG}(2, q^3)$  that is tangent to  $\ell_\infty$ . The order- $q$ -subplane  $\text{PG}(2, q)$  of  $\text{PG}(2, q^3)$  meets  $\ell_\infty$  in  $q + 1$  points. By [2, Lemma 2.6], we can map  $\text{PG}(2, q)$  to an order- $q$ -subplane  $\mathcal{B}$  tangent to  $\ell_\infty$  using the homography  $\zeta$  with matrix  $A_1$ , where

$$A_1 = \begin{pmatrix} -\tau & 1 + \tau & 0 \\ 0 & 1 & 0 \\ 0 & 1 + \tau & -\tau \end{pmatrix}, \quad A'_1 = \begin{pmatrix} -1 & 1 + \tau & 0 \\ 0 & \tau & 0 \\ 0 & 1 + \tau & -1 \end{pmatrix}.$$

The matrix  $A'_1$  is the matrix of the inverse homography  $\zeta^{-1}$  that maps  $\mathcal{B}$  to  $\text{PG}(2, q)$ .

Note that  $\zeta$  fixes the points  $T = (1, 0, 0)$  and  $P_\infty = (1, 1, 1)$  of  $\text{PG}(2, q)$ . The lines of  $\text{PG}(2, q)$  through  $T$  can be written with homogeneous coordinates  $m'_e = [0, 1 - e, e]$ ,  $e \in \text{GF}(q)$  and  $m'_\infty = TP_\infty = [0, 1, -1]$ . For  $e \in \text{GF}(q)$ , the line  $m'_e$  is mapped by  $\zeta$  to the line  $m_e = [0, e + \tau, -e]$  of  $\mathcal{B}$  (a line through  $T$ ). The line  $m'_\infty$  of  $\text{PG}(2, q)$  is fixed by  $\zeta$  and so  $m_\infty = [0, 1, -1]$  is the final line of  $\mathcal{B}$  through  $T$ .

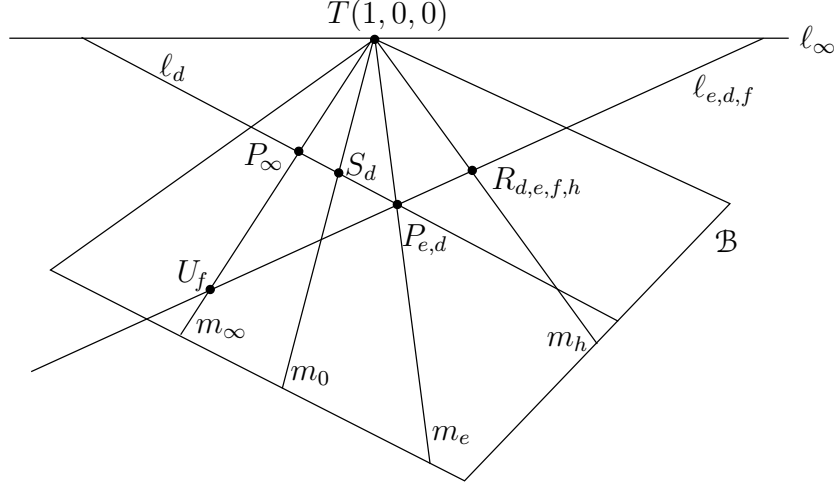


Figure 1: Tangent subplane notation

Next we find the coordinates of the points in  $\mathcal{B}$ , see Figure 1 for an illustration of the notation used. First consider the line  $m'_\infty$  of  $\text{PG}(2, q)$ . For  $f \in \text{GF}(q)$ , the point  $U'_f = (1 - f, 1, 1) \in m'_\infty$  is mapped under  $\zeta$  to the point  $U_f = (1 + f\tau, 1, 1)$  on  $m_\infty$  in  $\mathcal{B}$ . Now consider the line  $m'_0$  of  $\text{PG}(2, q)$ , it has points  $T$  and  $S'_d = (d, 0, 1)$ ,  $d \in \text{GF}(q)$ . The point  $S'_d$  maps under  $\zeta$  to  $S_d = (d, 0, 1)$ . So the affine points of  $m_0$  in  $\mathcal{B}$  are  $S_d$ ,  $d \in \text{GF}(q)$ .

We work out the coordinates of the other points of  $\mathcal{B}$  as follows. First define the  $q$  lines  $\ell_d = P_\infty S_d$ ,  $d \in \text{GF}(q)$ . Now for  $d, e, f, h \in \text{GF}(q)$ , let  $P_{e,d} = \ell_d \cap m_e$ ,  $\ell_{e,d,f} = P_{e,d} U_f$ , and  $R_{d,e,f,h} = \ell_{e,d,f} \cap m_h$ ,  $R_{e,d,f,\infty} = \ell_{e,d,f} \cap m_\infty = U_f$ . Working out the coordinates for these points and lines is straightforward, and they are given in Table 1.

The notation  $\ell_{e,d,f}$  is needed in subsequent calculations. However, if we wish to enumerate the lines of  $\mathcal{B}$ , we can use the following lemma.

**Lemma 6.1** *Every line in  $\mathcal{B}$  not through  $T$  is represented by some  $\ell_{0,d,f}$  with  $d, f \in \text{GF}(q)$ .*

**Proof** Consider the image of the general affine line  $t' = [1, a, b]$  of  $\text{PG}(2, q)$  not through  $T = (1, 0, 0)$ . Under  $\zeta$  it is mapped to  $t = [-1, 1 + b + (1 + a + b)\tau, -b]$ . Calculating  $\ell_{e,d,f}$  with  $e = 0$ ,  $d = -b$ ,  $f = 1 + a + b$  gives  $[-1, 0 \times f + b + 1 + (1 + a + b)\tau, -b]$  which is the line  $t$  as required.  $\square$

We can use this to calculate the coordinates of the tangent splash of  $\mathcal{B}$ .

**Lemma 6.2** *The tangent splash  $\mathcal{S}_T$  for  $\mathcal{B}$  has centre  $T = (1, 0, 0)$  and points  $V_{d,f} = \ell_{0,d,f} \cap \ell_\infty = (1 - d + f\tau, 1, 0)$  for  $d, f \in \text{GF}(q)$ . Alternatively, we can write  $\mathcal{S}_T = \{(a + b\tau, 1, 0) \mid a, b \in \text{GF}(q)\} \cup \{(1, 0, 0)\}$ .*

Table 1: Coordinates in the tangent order- $q$ -subplane  $\mathcal{B}$ ,  $(e, d, f, h \in \text{GF}(q))$

Notation	Coordinates	Description
$T$	$(1, 0, 0)$	$\mathcal{B} \cap \ell_\infty$
$P_\infty$	$(1, 1, 1)$	
$m_e$	$[0, e + \tau, -e]$	lines of $\mathcal{B}$ through $T$
$m_\infty$	$[0, 1, -1]$	$TP_\infty$
$S_d$	$(d, 0, 1)$	points of $\mathcal{B}$ on $m_0$
$U_f$	$(1 + f\tau, 1, 1)$	points of $\mathcal{B}$ on $m_\infty$
$\ell_d$	$[1, d - 1, -d]$	$P_\infty S_d$
$P_{e,d}$	$(e + d\tau, e, e + \tau)$	$m_e \cap \ell_d$
$\ell_{e,d,f}$	$[-1, ef - d + 1 + f\tau, d - ef]$	$P_{e,d}U_f$
$R_{e,d,f,h}$	$(h + (fh - fe + d)\tau, h, h + \tau)$	$\ell_{e,d,f} \cap m_h$
$R_{e,d,f,\infty}$	$(1 + f\tau, 1, 1)$	$\ell_{e,d,f} \cap m_\infty = U_f$

**Proof** By Lemma 6.1, the lines of  $\mathcal{B}$  are  $\ell_{0,d,f}$ . Let  $V_{d,f} = \ell_{0,d,f} \cap \ell_\infty = (1 - d + f\tau, 1, 0)$ , so the splash of  $\mathcal{B}$  is  $\mathcal{S}_T = \{T\} \cup \{V_{d,f} \mid f, d \in \text{GF}(q)\}$ . It is straightforward to rewrite this as  $\{(a + b\tau, 1, 0) \mid a, b \in \text{GF}(q)\} \cup \{(1, 0, 0)\}$ .  $\square$

## 7 Sublines of a tangent splash revisited

In this section we show that every order- $q$ -subline contained in a tangent splash must contain the centre. Further, we show that a tangent splash has a unique centre.

**Theorem 7.1** *Let  $\mathcal{S}_T$  be a tangent splash of  $\text{PG}(2, q^3)$ . Then every order- $q$ -subline contained in  $\mathcal{S}_T$  contains  $T$ .*

**Proof** By Corollary 4.2, the collineation group fixing  $\ell_\infty$  is transitive on the tangent splashes of  $\ell_\infty$ . So without loss of generality, we prove the result for the tangent splash of the order- $q$ -subplane  $\mathcal{B}$  coordinatised in Section 6. So using Lemma 6.2, we assume that  $\mathcal{S}_T = \{(d + f\tau, 1, 0) \mid d, f \in \text{GF}(q)\} \cup \{(1, 0, 0)\}$  with centre  $T = (1, 0, 0)$ . Let  $U = (0, 1, 0)$ ,  $V = (\alpha_1, 1, 0)$ ,  $W = (\alpha_2, 1, 0)$ , where  $\alpha_i = d_i + f_i\tau$ ,  $d_i, f_i \in \text{GF}(q)$ ,  $i = 1, 2$ , be three distinct non-centre points of  $\mathcal{S}_T$ , so  $\alpha_1 \neq \alpha_2$  and  $\alpha_1, \alpha_2 \neq 0$ . Let  $\ell$  be the unique order- $q$ -subline containing  $U, V, W$ , and suppose  $\ell$  is contained in  $\mathcal{S}_T$ . We will show that  $\ell$  contains  $T$ . As the collineation group fixing the tangent splash is transitive on the non-centre points of the tangent splash (by Lemma 4.4(7)) this is sufficient to show that all the order- $q$ -sublines contained in  $\mathcal{S}_T$  must contain  $T$ .

Consider the order- $q$ -subline  $\ell_0 = \{(a, 1, 0) \mid a \in \text{GF}(q)\} \cup \{(0, 1, 0)\}$  of  $\ell_\infty$ . To show that  $\ell$  contains  $T$ , we find a homography  $\phi$  that maps  $\ell_0$  to  $\ell$ , and demonstrate a point of  $\ell_0$  that  $\phi$  maps to  $T$ .

For brevity, we omit the last (zero) coordinate, and represent points by column vectors. Consider the matrix

$$\begin{pmatrix} 0 & \alpha_1\alpha_2 \\ \alpha_1 - \alpha_2 & \alpha_2 \end{pmatrix}.$$

It has nonzero determinant by the choice of  $\alpha_1, \alpha_2$ . Hence it represents a homography  $\phi$  which maps the points  $(1, 0)^t, (0, 1)^t, (1, 1)^t$  of  $\ell_0$  to  $U, V, W$  of  $\ell$  respectively, hence  $\phi$  maps the order- $q$ -subline  $\ell_0$  to the order- $q$ -subline  $\ell$ . So  $\phi$  maps the point  $(a, 1)^t$  of  $\ell_0$  to the point  $W_a = (\alpha_1\alpha_2, a(\alpha_1 - \alpha_2) + \alpha_2)^t$  of  $\ell$ . By assumption,  $W_a \in \mathcal{S}_T$ , so  $\alpha_1\alpha_2/(a(\alpha_1 - \alpha_2) + \alpha_2)$  is of the form  $d + f\tau$  for some  $d, f \in \text{GF}(q)$ . Now  $\alpha_1\alpha_2 = d_1d_2 + (d_1f_2 + d_2f_1)\tau + f_1f_2\tau^2$  and  $a(\alpha_1 - \alpha_2) + \alpha_2 = (a(d_1 - d_2) + d_2) + (a(f_1 - f_2) + f_2)\tau$ , so  $(\alpha_1\alpha_2)/(a(\alpha_1 - \alpha_2) + \alpha_2) = d + f\tau$  implies that

$$d_1d_2 + (d_1f_2 + d_2f_1)\tau + f_1f_2\tau^2 = (d + f\tau)((a(d_1 - d_2) + d_2) + (a(f_1 - f_2) + f_2)\tau).$$

Hence, for all  $a \in \text{GF}(q)$ ,

$$\begin{aligned} 0 &= [d(a(d_1 - d_2) + d_2) - d_1d_2] \\ &\quad + [f(a(d_1 - d_2) + d_2) + (a(f_1 - f_2) + f_2)d - (d_1f_2 + d_2f_1)]\tau \\ &\quad + [f(a(f_1 - f_2) + f_2) - f_1f_2]\tau^2. \end{aligned}$$

The constant term and the coefficient of  $\tau^2$  give

$$d = \frac{d_1d_2}{a(d_1 - d_2) + d_2} \quad \text{and} \quad f = \frac{f_1f_2}{a(f_1 - f_2) + f_2},$$

and substituting into the coefficient of  $\tau$  gives (where  $D = a(d_1 - d_2) + d_2$ ,  $F = a(f_1 - f_2) + f_2$ )

$$\begin{aligned} 0 &= \frac{f_1f_2}{F}D + \frac{d_1d_2}{D}F - (d_1f_2 + d_2f_1) \\ 0 &= (f_1D - d_1F)(f_2D - d_2F). \end{aligned}$$

Hence either  $(f_1D - d_1F) = 0$  or  $(f_2D - d_2F) = 0$ . Expanding and rearranging the first condition gives  $0 = a(f_1d_2 - f_2d_1) + (d_1f_2 - f_1d_2)$  for all  $a \in \text{GF}(q)$  and so  $f_1d_2 - f_2d_1 = 0$ . Expanding and rearranging the second condition gives  $0 = a(f_2d_1 - f_1d_2)$  for all  $a \in \text{GF}(q)$  and so we have the same condition  $f_1d_2 - f_2d_1 = 0$ .

We study the condition  $f_1d_2 - f_2d_1 = 0$  by considering the following four cases that cover all the possibilities: (1)  $d_2, f_2 \neq 0$ , (2)  $d_1 = d_2 = 0$ , (3)  $f_2 = 0, f_1 = 0$ , (4)  $d_2 = f_2 = 0$ . For case (1) we have  $f_1/f_2 = d_1/d_2$ , let  $k = f_1/f_2$ , so  $k \in \text{GF}(q)$ . Note that  $k \neq 1$  since otherwise  $V = W$ , contradicting  $U, V, W$  being distinct. Similarly,  $k \neq 0$ . So  $f_1 = kf_2$ ,  $d_1 = kd_2$ , thus  $\alpha_1 = k\alpha_2$ , and the homography  $\phi$  has matrix

$$\begin{pmatrix} 0 & k\alpha_2 \\ k - 1 & 1 \end{pmatrix},$$

so  $\phi$  maps the point  $(-1/(k - 1), 1)^t$  of  $\ell_0$  to  $T$ , that is,  $\ell$  contains  $T$ . For case (2), if  $f_1 = 0$  or  $f_2 = 0$ , then  $V = U$  or  $W = U$ , a contradiction. So  $f_1f_2 \neq 0$ , and  $\phi$  has matrix

$$\begin{pmatrix} 0 & f_1f_2\tau \\ f_1 - f_2 & f_2 \end{pmatrix},$$



so  $\phi$  maps the point  $(-f_2/(f_1 - f_2), 1)^t$  of  $\ell_0$  to  $T$ , so  $\ell$  contains  $T$ . For case (3), if  $d_1 = 0$  or  $d_2 = 0$ , then  $V = U$  or  $W = U$ , a contradiction. So  $d_1 d_2 \neq 0$ , and  $\phi$  has matrix

$$\begin{pmatrix} 0 & d_1 d_2 \\ d_1 - d_2 & d_2 \end{pmatrix},$$

so  $\phi$  maps the point  $(-d_2/(d_1 - d_2), 1)^t$  of  $\ell_0$  to  $T$ , so  $\ell$  contains  $T$ . Finally for case (4), if  $d_2 f_2 = 0$ , then  $U = W$ , a contradiction.

Hence in every case, the homography  $\phi$  maps the order- $q$ -subline  $\ell_0$  to the order- $q$ -subline  $\ell$  which contains  $U, V, W$  and also  $T$ . Thus every order- $q$ -subline through three points of  $\mathcal{S}_T$  that is contained in  $\mathcal{S}_T$  also contains the point  $T$ .  $\square$

We can use this result to show that a tangent splash has only one centre.

**Theorem 7.2** *A tangent splash has a unique centre.*

**Proof** We note that from Theorem 7.1, every order- $q$ -subline contained in a tangent splash  $\mathcal{S}_T$  with centre  $T$  contains  $T$ . Suppose  $\mathcal{S}_T$  has another centre  $U$ . Then every order- $q$ -subline contained in  $\mathcal{S}_T$  contains  $T$  and  $U$ , hence there are exactly  $(q^2 - 1)/(q - 1) = q + 1$  order- $q$ -sublines contained in  $\mathcal{S}_T$ , contradicting Lemma 3.1(2) which shows that there are  $q(q + 1)$  order- $q$ -sublines contained in  $\mathcal{S}_T$  which contain  $T$ .  $\square$

## 8 Tangent splashes and Sherk surfaces

We can uniquely identify points  $(a_0, a_1, a_2, 1)$  of  $\text{AG}(3, q)$ ,  $a_0, a_1, a_2 \in \text{GF}(q)$ , with elements  $a_0 + a_1\tau + a_2\tau^2$  of  $\text{GF}(q^3)$ . In [13], Sherk considers this representation of  $\mathcal{A} = \text{AG}(3, q)$  and lets  $\overline{\mathcal{A}} = \text{AG}(3, q) \cup \{\infty\}$ . A *Sherk surface*  $\mathcal{S}(\theta, K, L, \phi)$ ,  $\theta, \phi \in \text{GF}(q)$ ,  $K, L \in \text{GF}(q^3)$ , is the set of points  $X$  in  $\overline{\mathcal{A}}$  satisfying

$$\theta X^{1+q+q^2} + (KX^{1+q} + K^q X^{q+q^2} + K^{q^2} X^{1+q^2}) + (LX + L^q X^q + L^{q^2} X^{q^2}) + \phi = 0.$$

There are four orbits of Sherk surfaces, uniquely determined by the size of the surfaces in each orbit. A Sherk surface has size 1,  $q^2 - q + 1$ ,  $q^2 + 1$ , or  $q^2 + q + 1$ . The Sherk surfaces of size  $q^2 + q + 1$  are precisely the Bruck covers/hyper-reguli, see [12]. In future work, we show that an *exterior splash* of  $\text{PG}(2, q^3)$  (the splash of an order- $q$ -subplane exterior to  $\ell_\infty$ ) is a Sherk surface of size  $q^2 + q + 1$ . We show here that a tangent splash of  $\text{PG}(2, q^3)$  is a Sherk surface of size  $q^2 + 1$ .

**Theorem 8.1** *A tangent splash of  $\text{PG}(2, q^3)$  is a Sherk surface of size  $q^2 + 1$ .*

**Proof** We can naturally identify the points of  $\ell_\infty = \text{PG}(1, q^3)$  with elements of  $\text{GF}(q^3) \cup \{\infty\}$ : the point  $(a, 1, 0) \in \ell_\infty$  corresponds to the element  $a \in \text{GF}(q^3)$ , and the point

$(1, 0, 0)$  corresponds to the element  $\infty$ . By Corollary 4.2 and Lemma 6.2, we can without loss of generality, take our tangent splash to be  $\mathcal{S}_T = \{(a + b\tau, 1, 0) \mid a, b \in \text{GF}(q)\} \cup \{(1, 0, 0)\}$ . The four points  $(1, 0, 0), (0, 1, 0), (1, 1, 0), (\tau, 1, 0)$  are contained in  $\mathcal{S}_T$ , and are not on an order- $q$ -subline, so by Theorem 5.5, they uniquely determine  $\mathcal{S}_T$ .

Sherk shows that a plane of  $\mathcal{A} = \text{AG}(3, q)$  together with  $\infty$  is a Sherk surface  $\mathcal{S}(0, 0, L, \phi)$ ,  $L \in \text{GF}(q^3)$ ,  $\phi \in \text{GF}(q)$  of size  $q^2 + 1$ . This Sherk surface is the set of points satisfying

$$(LX + L^q X^q + L^{q^2} X^{q^2}) + \phi = 0.$$

We show that our tangent splash is one of these planes. The centre of our tangent splash  $\mathcal{S}_T$  is the point  $T = (1, 0, 0)$ , which corresponds to the element  $\infty$ , which is on every Sherk surface that corresponds to a plane in  $\text{AG}(3, q)$ . The point  $(0, 1, 0)$  corresponds to the element 0 of  $\text{GF}(q^3)$ . This is on the Sherk surface  $\mathcal{S}(0, 0, L, \phi)$  if and only if  $\phi = 0$ . Now consider the points  $(1, 1, 0), (\tau, 1, 0)$  of  $\mathcal{S}_T$  which correspond to the elements 1,  $\tau$  of  $\text{GF}(q^3)$  respectively. These lie on the Sherk surface  $\mathcal{S}(0, 0, L, 0)$  if

$$L + L^q + L^{q^2} = 0, \tag{2}$$

$$L\tau + L^q \tau^q + L^{q^2} \tau^{q^2} = 0. \tag{3}$$

These equations have a unique nonzero solution for  $L$ , since in  $\text{AG}(3, q)$  there is a unique plane through the three points 0, 1,  $\tau$ , hence there is a unique Sherk surface  $\mathcal{S}(0, 0, L, \phi)$  containing these three points. Note this Sherk surface also contains  $\infty$ .

Now  $a \times (2) + b \times (3)$  gives:  $L(a + b\tau) + L^q(a + b\tau^q) + L^{q^2}(a + b\tau^{q^2}) = 0$ . If  $a, b \in \text{GF}(q)$ , then this can be written as

$$L(a + b\tau) + L^q(a + b\tau)^q + L^{q^2}(a + b\tau)^{q^2} = 0.$$

Hence the element  $a + b\tau$  of  $\text{GF}(q^3)$  lies on the Sherk surface  $\mathcal{S}(0, 0, L, 0)$ . That is, the Sherk surface containing the elements  $\infty, 0, 1, \tau$  contains all the elements  $a + b\tau$ , where  $a, b \in \text{GF}(q)$ . So this Sherk surface is equivalent to our tangent splash  $\mathcal{S}_T = \{(a + b\tau, 1, 0) \mid a, b \in \text{GF}(q)\} \cup \{(1, 0, 0)\}$ .  $\square$

## 9 Cover planes of a splash in $\text{PG}(5, q)$

In this section we look at the points of a tangent splash in the Bruck-Bose representation in  $\text{PG}(6, q)$ . A tangent splash  $\mathcal{S}_T$  of  $\ell_\infty$  in  $\text{PG}(2, q^3)$  corresponds in  $\text{PG}(6, q)$  to a set  $[\mathcal{S}_T]$  of  $q^2 + 1$  planes of the spread  $\mathcal{S}$  in  $\Sigma_\infty \cong \text{PG}(5, q)$ . We show that there is an interesting set of planes in  $\text{PG}(5, q)$  meeting every element of the splash and contained entirely within the splash.

**Theorem 9.1** *Let  $\mathcal{S}_T$  be a tangent splash of  $\ell_\infty$  with centre  $T$ , and let  $[\mathcal{S}_T]$  be the corresponding set of planes in  $\Sigma_\infty$  in the Bruck-Bose representation in  $\text{PG}(6, q)$ . There are*

exactly  $q^2 + q + 1$  planes of  $\Sigma_\infty \cong \text{PG}(5, q)$  that meet every plane of  $[\mathcal{S}_T]$ , called **cover planes**. The cover planes each meet the centre  $[T]$  in distinct lines, and meet every other plane of  $[\mathcal{S}_T]$  in distinct points, and hence are contained entirely within the splash.

**Proof** By Corollary 4.2, without loss of generality, we can let  $\mathcal{S}_T$  be the tangent splash determined by the order- $q$ -subplane  $\mathcal{B}$  coordinatised in Section 6. So by Lemma 6.2,  $\mathcal{S}_T = \{T = (1, 0, 0)\} \cup \{X_{a,b} = (a + b\tau, 1, 0) \mid a, b \in \text{GF}(q)\}$ . In the Bruck-Bose representation in  $\text{PG}(6, q)$ , the plane  $[X_{a,b}]$  consists of the points with homogeneous coordinates  $\sigma(\delta(a + b\tau), \delta, 0)$  for  $\delta \in \text{GF}(q^3)$ , where  $\sigma$  is the generalised Bruck-Bose map defined in Section 2.1. Consider the plane  $\pi = \langle \sigma(\tau, 0, 0), \sigma(0, \tau, 0), \sigma(\tau^2, 0, 0) \rangle$  of  $\Sigma_\infty$ . The plane  $\pi$  meets  $[T]$  in the line  $\langle \sigma(\tau, 0, 0), \sigma(\tau^2, 0, 0) \rangle$  and meets the plane  $[X_{a,b}]$ ,  $a, b \in \text{GF}(q)$ , in the point  $A_{a,b} = \sigma(a\tau + b\tau^2, \tau, 0)$ . That is,  $\pi$  meets  $[T]$  in a line and meets every other plane of  $[\mathcal{S}_T]$  in a point, this accounts for all the points of  $\pi$ . Recall the mapping  $\Theta$  from Theorem 2.3, the group  $\langle \Theta \rangle$  fixes the planes of the spread  $\mathcal{S}$  and acts regularly on the points and lines of these planes. Hence  $\Theta$  maps  $\pi$  to a plane that meets  $[T]$  in a line, and meets the other planes of  $[\mathcal{S}_T]$  in points, so  $\Theta$  maps  $\pi$  to another cover plane of  $[\mathcal{S}_T]$ . As  $\langle \Theta \rangle$  acts regularly on the lines of  $[T]$ , there are at least  $q^2 + q + 1$  cover planes.

By Lemma 5.2(3), the number of tangent splashes with a common centre  $[T]$  is  $q(q^2 + q + 1)$ . Each of these tangent splashes has at least  $q^2 + q + 1$  cover planes, giving at least  $q(q^2 + q + 1)^2$  distinct planes of  $\Sigma_\infty$  that meet  $[T]$  in a line. As there are in total  $q(q^2 + q + 1)^2$  planes of  $\text{PG}(5, q)$  that meet  $[T]$  in a line, every plane that meets  $[T]$  in a line is a cover plane for some tangent splash with centre  $[T]$ . Further, if a plane  $\pi$  meets one element  $[P]$  of a tangent splash  $[\mathcal{S}_T]$  in a line, and every other element of  $[\mathcal{S}_T]$  in a point, then  $\pi$  must be a cover plane for a splash with centre  $[P]$ . As a splash has a unique centre by Theorem 7.2,  $[P]$  is the centre of the splash  $[\mathcal{S}_T]$ , that is,  $P = T$ . Hence a tangent splash has exactly  $q^2 + q + 1$  cover planes, each meeting the centre in a line. Note also that if  $\pi$  is a plane of  $\text{PG}(5, q)$  that meets a spread element  $[T]$  in a line, then the planes of  $\mathcal{S}$  that meet  $\pi$  form a tangent splash with centre  $[T]$ .  $\square$

As a direct consequence of this proof we have the following properties of cover planes.

**Corollary 9.2** *The  $q^2 + q + 1$  cover planes of a tangent splash  $[\mathcal{S}_T]$  lie in an orbit of  $\langle \Theta \rangle$ .*

**Corollary 9.3** *If a plane of  $\Sigma_\infty$  meets  $q^2 + 1$  planes of the spread  $\mathcal{S}$ , then these  $q^2 + 1$  planes form a tangent splash.*

**Corollary 9.4** *Two distinct cover planes of a tangent splash  $[\mathcal{S}_T]$  meet in exactly one point, which lies in the centre  $[T]$ .*

**Proof** Recall that in the proof of Theorem 9.1, we constructed the  $q^2 + q + 1$  cover planes by considering the action of  $\Theta$  on the tangent splash calculated in Lemma 6.2. As  $\langle \Theta \rangle$  acts regularly on the points of the spread element  $[X_{a,d}]$ , the  $q^2 + q + 1$  cover planes meet

$[X_{a,d}]$  in distinct points. So any two cover planes meet in exactly one point which is in  $[T]$ .  $\square$

The next two theorems contain further properties of cover planes which will be useful later.

**Theorem 9.5** *Let  $[\mathcal{S}_T]$  be a tangent splash in the Bruck-Bose representation of  $\text{PG}(2, q^3)$  in  $\text{PG}(6, q)$ . Let  $\ell$  be a line that meets  $q+1$  elements of  $[\mathcal{S}_T]$ , then  $\ell$  lies in a unique cover plane of the splash  $[\mathcal{S}_T]$ .*

**Proof** As in the proof of Theorem 9.1, without loss of generality let  $\mathcal{S}_T = \{T = (1, 0, 0)\} \cup \{X_{a,b} = (a + b\tau, 1, 0) \mid a, b \in \text{GF}(q)\}$ . Let  $\ell$  be a line that meets  $q+1$  elements of  $[\mathcal{S}_T]$ . As  $\mathcal{S}$  is a regular spread, these  $q+1$  splash elements form a 2-regulus. By Theorem 2.1,  $\ell$  corresponds in  $\text{PG}(2, q^3)$  to an order- $q$ -subline  $\ell$  of  $\ell_\infty$ . By Theorem 7.1,  $\ell$  contains the centre  $T$ , so in  $\text{PG}(6, q)$ , the line  $\ell$  meets the centre  $[T]$ . By Lemma 4.4(7) we can assume without loss of generality that the order- $q$ -subline  $\ell$  in  $\ell_\infty$  contains the point  $V = (0, 1, 0)$ . Let  $W = (d + f\tau, 1, 0)$  for some  $d, f \in \text{GF}(q)$  be another point on  $\ell$ . So in  $\text{PG}(6, q)$ , the line  $\ell$  meets the splash elements  $[T]$ ,  $[V]$  and  $[W]$ . By Theorem 2.3, we may assume  $\ell$  meets  $[V]$  in the point  $C = \sigma(0, 1, 0)$ . There is a unique line through  $C$  that meets  $[T]$  and  $[W]$ , so we can calculate that  $\ell$  meets  $W$  in the point  $\sigma(d + f\tau, 1, 0)$  and hence meets  $[T]$  in the point  $A_\infty = \sigma(d + f\tau, 0, 0)$ .

Suppose  $d \neq 0$ . Let  $A'_\infty = \sigma(\tau, 0, 0)$  and consider the plane  $\pi = \langle A_\infty, A'_\infty, C \rangle$ , it meets the splash element  $[X_{a,b}]$  ( $a, b \in \text{GF}(q)$ ) in the point  $\sigma(a + b\tau, 1, 0) = (a/d)A_\infty + ((bd - af)/d)A'_\infty + C$ . Thus  $\pi$  is a cover plane of  $[\mathcal{S}_T]$  containing the line  $\ell$ . By Corollary 9.4, cover planes do not meet off  $[T]$ , hence there is exactly one cover plane containing  $\ell$ .

Now suppose that  $d = 0$ , so  $f \neq 0$  as  $W \neq T$ . In this case let  $A'_\infty = \sigma(1, 0, 0)$  and let  $\pi = \langle A_\infty, A'_\infty, C \rangle$ . Then  $\pi$  meets the spread element  $[X_{a,b}]$  ( $a, b \in \text{GF}(q)$ ) in the point  $\sigma(a + b\tau, 1, 0) = (a/f)A_\infty + aA'_\infty + C$ , and so  $\pi$  is the required cover plane.  $\square$

**Theorem 9.6** *Let  $[\mathcal{S}_T]$  be a tangent splash in the Bruck-Bose representation of  $\text{PG}(2, q^3)$  in  $\text{PG}(6, q)$ . Let  $\pi$  be a plane of  $\Sigma_\infty = \text{PG}(5, q)$  which meets the centre  $[T]$  in a line, and meets three further elements  $[U], [V], [W]$  of  $[\mathcal{S}_T]$ , where  $[T], [U], [V], [W]$  are not in a common 2-regulus. Then  $\pi$  is a cover plane of  $[\mathcal{S}_T]$ .*

**Proof** Let  $\pi$  be a plane that meets the centre  $[T]$  in a line and meets three further elements  $[U], [V], [W]$  of  $[\mathcal{S}_T]$  at points  $U', V', W'$  respectively, where  $[T], [U], [V], [W]$  are not in a common 2-regulus. Then  $U'V' \cap [T], U', V'$  are three points on a line  $m$  of  $\pi$ . As  $m$  meets the splash elements  $[T], [U], [V]$ , by Lemma 3.1(2) the line  $m$  is contained entirely within the splash. Similarly  $U'W' \cap [T], U', W'$  define a line  $n$  of  $\pi$  that lies entirely in the splash. Note that  $n \neq m$  as  $[T], [U], [V], [W]$  are not in a common 2-regulus. Repeating this argument for the remaining lines of  $\pi$  which join a point of  $m$  to a point of  $n$  shows that  $\pi$  is contained entirely within the splash. That is,  $\pi$  is a cover plane of the splash.  $\square$

## 10 The tangent subspace at an affine point

Consider an order- $q$ -subplane  $\mathcal{B}$  in  $\text{PG}(2, q^3)$  tangent to  $\ell_\infty$  at the point  $T$ . For each affine point  $P$  of  $\mathcal{B}$ , we construct an order- $q$ -subplane  $P^\perp$  that contains  $P$  and is secant to  $\ell_\infty$  as follows. Let  $\ell_1, \dots, \ell_{q+1}$  be the  $q+1$  lines of  $\mathcal{B}$  through  $P$ . Then by Lemma 3.1(1),  $m = \{\ell_i \cap \ell_\infty \mid i = 1, \dots, q+1\}$  is an order- $q$ -subline of  $\ell_\infty$  through  $T$ . Now  $m$  and  $PT \cap \mathcal{B}$  are two order- $q$ -sublines through  $T$ , and so lie in a unique order- $q$ -subplane which we denote by  $P^\perp$ . We show in Corollary 10.3 that the order- $q$ -subplanes  $\mathcal{B}$  and  $P^\perp$  meet in exactly the order- $q$ -subline  $PT \cap \mathcal{B}$ .

By Theorem 2.2, in  $\text{PG}(6, q)$ ,  $\mathcal{B}$  corresponds to a ruled surface  $[\mathcal{B}]$  and  $P^\perp$  corresponds to a plane  $[P^\perp]$  that meets  $q+1$  elements of the spread  $\mathcal{S}$ . We will show in Theorem 10.2 that  $[P^\perp]$  is the tangent space to  $[\mathcal{B}]$  at the point  $P$ . This leads us to call  $P^\perp$  the *tangent space* of  $P$ .

By Theorem 4.1 we can without loss of generality let  $\mathcal{B}$  be the order- $q$ -subplane coordinatised in Section 6. Consider the point  $P_{e,d}$  of  $\mathcal{B}$  for some  $e, d \in \text{GF}(q)$  and consider the order- $q$ -subplane  $P_{e,d}^\perp$ . We work in  $\text{PG}(6, q)$  and in the next lemma determine the coordinates of two points in  $[P_{e,d}^\perp] \cap \Sigma_\infty$ . We use the following notation for  $e \in \text{GF}(q)$ :

$$\begin{aligned}\theta^7(e) &= (e + \tau)^{q^2+q} = (e + \tau^q)(e + \tau^{q^2}), \\ \theta(e) &= (e + \tau)^{q^2+q+1} = (e + \tau)(e + \tau^q)(e + \tau^{q^2}).\end{aligned}$$

Note that since  $\theta(e)^q = \theta(e)$ , it follows that  $\theta(e) \in \text{GF}(q)$ . We again use the generalised Bruck-Bose map  $\sigma$  defined in Section 2.1.

**Lemma 10.1** *In  $\text{PG}(6, q)$ , the plane  $[P_{e,d}^\perp]$ ,  $d, e \in \text{GF}(q)$ , contains the points  $J_{e,d}$ ,  $K_{e,d}$  of  $\Sigma_\infty$ , where*

$$J_{e,d} = \sigma((1-d)\tau\theta^7(e)^2, \tau\theta^7(e)^2, 0), \quad K_{e,d} = \sigma((1-d+e+\tau)\tau\theta^7(e)^2, \tau\theta^7(e)^2, 0).$$

**Proof** In  $\text{PG}(2, q^3)$ , the order- $q$ -subplane  $P_{e,d}^\perp$  contains all the affine points of the order- $q$ -subline  $m_e = TP_{e,d} \cap \mathcal{B}$ . Firstly, suppose that  $d \neq 0$ , so  $P_{e,d}$  and  $P_{e,0}$  are distinct points of  $m_e$ . Consider the two lines  $\ell_{e,d,0}$  and  $\ell_{e,d,1}$  through  $P_{e,d}$ . Using Table 1 we calculate  $V_{e,d,0} = \ell_{e,d,0} \cap \ell_\infty = (-d+1, 1, 0)$  and  $V_{e,d,1} = \ell_{e,d,1} \cap \ell_\infty = (e-d+1+\tau, 1, 0)$ . Now  $P_{e,d}, P_{e,0}, V_{e,d,0}, V_{e,d,1}$  are all distinct points of  $P_{e,d}^\perp$ , so  $X_{e,d} = P_{e,d}V_{e,d,1} \cup P_{e,0}V_{e,d,0}$  is in  $P_{e,d}^\perp$ . Straightforward calculation shows  $X_{e,d} = (e^2 + (e-d+d^2)\tau, e^2 + (e-d)\tau, (e+\tau)^2)$ . Note that the lines  $P_{e,0}X_{e,d}$ ,  $P_{e,d}X_{e,d}$  meet  $\ell_\infty$  in points of the order- $q$ -subplane  $P_{e,d}^\perp$ . The order- $q$ -subline of  $P_{e,d}^\perp$  through  $P_{e,d}$  and  $X_{e,d}$  contains the point  $V_{e,d,1}$  of  $\ell_\infty$ , so by Theorem 2.1, corresponds in  $\text{PG}(6, q)$  to the line through  $P_{e,d}$  and  $X_{e,d}$ . Hence in  $\text{PG}(6, q)$ , the point  $P_{e,d}X_{e,d} \cap \Sigma_\infty$  lies in the plane  $[P_{e,d}^\perp]$ . Similarly the point  $P_{e,0}X_{e,d} \cap \Sigma_\infty$  lies in  $[P_{e,d}^\perp]$ . To calculate the coordinates of these points in  $\text{PG}(6, q)$ , we need to take the coordinates of  $X_{e,d}, P_{e,0}, P_{e,d}$  in  $\text{PG}(2, q^3)$  and write them with third coordinate in  $\text{GF}(q)$ . So using the

Bruck-Bose map  $\sigma$ , we have in  $\text{PG}(6, q)$ ,

$$\begin{aligned} X_{e,d} &= \sigma((e^2 + (e - d + d^2)\tau)\theta^-(e)^2, (e^2 + (e - d)\tau)\theta^-(e)^2, \theta(e)^2), \\ P_{e,0} &= \sigma(e(e + \tau)\theta^-(e)^2, e(e + \tau)\theta^-(e)^2, \theta(e)^2), \\ P_{e,d} &= \sigma((e + d\tau)(e + \tau)\theta^-(e)^2, e(e + \tau)\theta^-(e)^2, \theta(e)^2). \end{aligned}$$

Now  $X_{e,d} = P_{e,0} - dJ_{e,d}$ , hence the line  $P_{e,0}X_{e,d}$  of  $\text{PG}(6, q)$  meets  $\Sigma_\infty$  in the point  $J_{e,d}$ , so  $J_{e,d}$  is in the plane  $[P_{e,d}^\perp]$ . Similarly,  $X_{e,d} = P_{e,d} - dK_{e,d}$ , hence the line  $P_{e,d}X_{e,d}$  of  $\text{PG}(6, q)$  meets  $\Sigma_\infty$  in the point  $K_{e,d}$ , so  $K_{e,d}$  is in the plane  $[P_{e,d}^\perp]$ .

Now suppose  $d = 0$ . In this case, let  $X_e = P_{e,1}V_{e,0,1} \cap P_{e,0}V_{e,0,0}$ . Straightforward calculation gives

$$\begin{aligned} X_e &= (e^2 + e\tau - \tau, e^2 + e\tau - \tau, (e + \tau)^2) \\ &= ((e^2 + e\tau - \tau)\theta^-(e)^2, (e^2 + e\tau - \tau)\theta^-(e)^2, \theta(e)^2). \end{aligned}$$

Then in  $\text{PG}(6, q)$ ,  $X_e = P_{e,0} - J_{e,0}$  and  $X_e = P_{e,1} - K_{e,0}$ . So  $J_{e,0}, K_{e,0} \in [P_{e,d}^\perp]$  as required.  $\square$

**Theorem 10.2** *Let  $\mathcal{B}$  be a tangent order- $q$ -subplane in  $\text{PG}(2, q^3)$  and let  $[\mathcal{B}]$  be the corresponding ruled surface in  $\text{PG}(6, q)$ . By Theorem 2.2,  $[\mathcal{B}]$  is the intersection of nine quadrics. Let  $P$  be an affine point of  $\mathcal{B}$ , then in  $\text{PG}(6, q)$ , the plane  $[P^\perp]$  is the intersection of the tangent spaces at  $P$  to each of these nine quadrics.*

**Proof** As before, without loss of generality we prove this for the tangent order- $q$ -subplane  $\mathcal{B}$  coordinatised in Section 6. The equations of the nine quadrics determining  $[\mathcal{B}]$  are given in [2] in equations (16), (17), (18), and are:

$$(1 - x)^q((1 + \tau)y - 1) - (1 - x)((1 + \tau)y - 1)^q = 0, \quad (4)$$

$$(1 - x)(\tau y)^q - (1 - x)^q \tau y = 0, \quad (5)$$

$$(1 - y)^q(1 - x) - (1 - y)(1 - x)^q = 0. \quad (6)$$

These equations are in  $\text{PG}(2, q^3)$ , and each equation corresponds to three quadrics in  $\text{PG}(6, q)$ . A point  $(x, y, 1)$  of  $\text{PG}(2, q^3)$  satisfying one of these equations corresponds to a point  $\sigma(x, y, 1)$  in  $\text{PG}(6, q)$  lying on all three quadrics. Note that equation (4) is equation (6) minus equation (5), so we only need consider the second two equations.

By Lemma 4.4(2), without loss of generality we can prove the result for the affine point  $P_{e,d}$  in  $\mathcal{B}$  for some  $e, d \in \text{GF}(q)$ . In  $\text{PG}(6, q)$  we want to find all lines through  $P_{e,d}$  tangent to each of the nine quadrics. A line  $\ell$  through the point  $P_{e,d}$  meets  $\Sigma_\infty$  in a point  $R = \sigma(u, v, 0)$  where  $u, v \in \text{GF}(q^3)$  depend on  $e, d$ . A general point on the line  $\ell$  has form  $P_{e,d} + tR$ , where  $t \in \text{GF}(q)$ . To determine which lines  $\ell$  are tangent lines, we substitute this general point into the equations for the quadrics and solve for points  $R$  where  $t = 0$  is a double root.

From Table 1 we have  $P_{e,d} = (e + d\tau, e, e + \tau)$ . Writing this as  $(x, y, 1)$  gives

$$x = (e + d\tau) \frac{\theta(e)}{\theta(e)}, \quad y = e \frac{\theta(e)}{\theta(e)}, \quad 1 - x = (1 - d)\tau \frac{\theta(e)}{\theta(e)}, \quad 1 - y = \tau \frac{\theta(e)}{\theta(e)}. \quad (7)$$

Replacing  $x$  with  $x + tu$  and  $y$  with  $y + tv$  in (5) we obtain

$$\begin{aligned} 0 &= (1 - x - tu)(\tau(y + tv))^q - (1 - x - tu)^q \tau(y + tv) \\ &= (1 - x)(\tau y)^q - (1 - x)^q \tau y + t^2(-u\tau^q v^q + u^q \tau v) + \\ &\quad + t(-u\tau^q y^q + (1 - x)\tau^q v^q - (1 - x)^q \tau v + u^q \tau y). \end{aligned}$$

Note that  $P_{e,d} = (x, y, 1)$  satisfies (5), so when the above equation is regarded as a polynomial in  $t$ , the constant term is equal to 0. For  $t = 0$  to be a repeated root, we need the coefficient of  $t$  to be equal to zero, that is  $-u\tau^q y^q + (1 - x)\tau^q v^q - (1 - x)^q \tau v + u^q \tau y = 0$ . Substituting for  $x, y$  using (7) yields

$$0 = -u\tau^q e \frac{\theta(e)^q}{\theta(e)} + (1 - d)\tau \frac{\theta(e)}{\theta(e)} \tau^q v^q - (1 - d)\tau^q \frac{\theta(e)^q}{\theta(e)} \tau v + u^q \tau e \frac{\theta(e)}{\theta(e)}.$$

Rearranging gives

$$\frac{1}{\tau\theta(e)} ((1 - d)\tau v + eu) = \left( \frac{1}{\tau\theta(e)} ((1 - d)\tau v + eu) \right)^q,$$

and so  $((1 - d)\tau v + eu) / \tau\theta(e)$  is in  $\text{GF}(q)$ . So we want  $u, v \in \text{GF}(q^3)$  such that  $((1 - d)\tau v + eu) / \tau\theta(e) = a$ , for any  $a \in \text{GF}(q)$ . That is,

$$(1 - d)\tau v + eu = a\tau\theta(e) \quad (8)$$

for  $a \in \text{GF}(q)$ . We now repeat these calculations for equation (6): we replace  $x$  by  $x + tu$  and  $y$  by  $y + tv$  to obtain

$$(1 - y - tv)^q (1 - x - tu) - (1 - y - tv)(1 - x - tu)^q = 0,$$

setting the coefficient of  $t$  to 0 gives

$$-v^q(1 - x) - (1 - y)^q u + (1 - y)u^q + v(1 - x)^q = 0,$$

and substituting for  $x, y$  from (7) gives  $((1 - d)v - u) / \tau\theta(e)$  is an element of  $\text{GF}(q)$ . So we want  $u, v \in \text{GF}(q^3)$  such that

$$(1 - d)v - u = b\tau\theta(e) \quad (9)$$

for any  $b \in \text{GF}(q)$ . Solving (8) and (9) for  $u, v$  gives

$$u = \frac{1}{\theta(e)}(a + be)\tau\theta(e)^2 - b\tau\theta(e), \quad v = \frac{1}{1 - d} \frac{1}{\theta(e)}(a + be)\tau\theta(e)^2,$$

where  $a, b$  are any values in  $\text{GF}(q)$ , that is,  $u, v$  are functions of  $a, b \in \text{GF}(q)$ . Hence the points  $R$  for which  $P_{e,d}R$  is a tangent line to the nine quadrics forming the variety  $[\mathcal{B}]$  are, for any  $a, b \in \text{GF}(q)$ ,

$$\begin{aligned} R &= \sigma(u, v, 0) \\ &= \left( \frac{a + be}{(1 - d)\theta(e)} \right) \sigma((1 - d)\tau\theta(e)^2, \tau\theta(e)^2, 0) - b\sigma(\tau\theta(e), 0, 0). \end{aligned}$$

Note that  $(a + be)/((1 - d)\theta(e)) \in \text{GF}(q)$ . By varying  $a, b \in \text{GF}(q)$ , we see that  $R$  is any point on the line joining  $J_{e,d} = \sigma((1 - d)\tau\theta(e)^2, \tau\theta(e)^2, 0)$  and  $C_e = \sigma(\tau\theta(e), 0, 0)$ . Hence the lines through  $P_{e,d}$  that are tangent to  $[\mathcal{B}]$  form a plane  $\alpha = \langle P_{e,d}, J_{e,d}, C_e \rangle$ . By Lemma 10.1, the point  $J_{e,d}$  is in the plane  $[P_{e,d}^\perp]$ . Further, the point  $K_{e,d} = J_{e,d} + \theta(e)C_e$  is in  $\alpha$  as  $\theta(e) \in \text{GF}(q)$ , and is in  $[P_{e,d}^\perp]$  by Lemma 10.1. Hence  $\alpha = [P_{e,d}^\perp]$ , that is, the plane  $[P_{e,d}^\perp]$  is formed from the lines tangent to  $[\mathcal{B}]$  at  $P_{e,d}$ .  $\square$

**Corollary 10.3** *Let  $\mathcal{B}$  be a tangent order- $q$ -subplane of  $\text{PG}(2, q^3)$  and let  $P$  be an affine point of  $\mathcal{B}$ . Then the order- $q$ -subplanes  $\mathcal{B}$  and  $P^\perp$  meet in exactly the order- $q$ -subline  $PT \cap \mathcal{B}$ . Further, there is a bijection from affine points  $P$  of  $\mathcal{B}$  to order- $q$ -subplanes  $P^\perp$  that contain the order- $q$ -subline  $PT \cap \mathcal{B}$  and meet  $\ell_\infty$  in an order- $q$ -subline contained in  $\mathcal{S}_T$ .*

**Proof** The proof of Theorem 10.2 shows that  $\mathcal{B}$  and  $P^\perp$  meet in the order- $q$ -subline  $PT \cap \mathcal{B}$ . The bijection follows from Lemma 3.1(2).  $\square$

The next corollary states a result from the final paragraph of the proof of Theorem 10.2. This result is needed in another article.

**Corollary 10.4**  $[P_{e,d}^\perp] = \langle P_{e,d}, J_{e,d}, C_e \rangle$ .

## 11 Conclusion

This article explored further properties of a tangent order- $q$ -subplane  $\mathcal{B}$  in  $\text{PG}(2, q^3)$ . In particular, we investigated the *tangent splash* of  $\mathcal{B}$  (the intersection of the lines of  $\mathcal{B}$  with  $\ell_\infty$ ). Most of this investigation was carried out in the plane  $\text{PG}(2, q^3)$ . However, the tangent splash has interesting properties when looked at in the Bruck-Bose representation of  $\text{PG}(2, q^3)$  in  $\text{PG}(6, q)$ . In particular, we demonstrated a set of cover planes, and investigated the tangent subspace of an affine point of  $\mathcal{B}$ . In another article [4], the authors continue this investigation by working in  $\text{PG}(6, q)$  and looking at the interaction between the ruled surface corresponding to  $\mathcal{B}$  and the set of planes forming the tangent splash of  $\mathcal{B}$ .



## References

- [1] J. André. Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe. *Math. Z.*, **60** (1954) 156–186.
- [2] S.G. Barwick and W.A. Jackson. Sublines and subplanes of  $\text{PG}(2, q^3)$  in the Bruck-Bose representation in  $\text{PG}(6, q)$ . *Finite Fields Th. App.* 18 (2012) 93–107.
- [3] S.G. Barwick and W.A. Jackson. A characterisation of tangent subplanes of  $\text{PG}(2, q^3)$ . To appear, *Designs Codes, Crypt.* (DOI) 10.1007/s10623-012-9754-7
- [4] S.G. Barwick and W.A. Jackson. The tangent splash in  $\text{PG}(6, q)$ . Submitted.
- [5] R.H. Bruck. Construction problems of finite projective planes. *Conference on Combinatorial Mathematics and its Applications*, University of North Carolina Press, (1969) 426–514.
- [6] R.H. Bruck and R.C. Bose. The construction of translation planes from projective spaces. *J. Algebra*, **1** (1964) 85–102.
- [7] R.H. Bruck and R.C. Bose. Linear representations of projective planes in projective spaces. *J. Algebra*, **4** (1966) 117–172.
- [8] L.R.A. Casse. *A Guide to Projective Geometry*. Oxford University Press, 2006.
- [9] J.W.P. Hirschfeld. *Finite Projective Spaces of Three Dimensions*. Oxford University Press, 1985.
- [10] J.W.P. Hirschfeld. *Projective Geometry over Finite Fields, Second Edition*. Oxford University Press, 1998.
- [11] J.W.P. Hirschfeld and J.A. Thas. *General Galois Geometries*. Oxford University Press, 1991.
- [12] N.L. Johnson, V. Jha and M. Biliotti. *Handbook of Finite Translation Planes*. Chapman and Hall/CRC 2007.
- [13] F.A. Sherk. The Geometry of  $\text{GF}(q^3)$ . *Can. J. Math.* 38 (1986) 672–696.